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The Circle will Now be Closed, Finally

(On the Inability to Adapt Circular Motion to Arbitrary Motions)

E. F. M. Jochim

German Aerospace Centre, Oberpfaffenhofen, Germany

Tel: (+49)-8153-28-2738, Fax: 0-8153-28-1452, e-mail: Fritz.Jochim@dlr.de

Summary: The description of the motion of the Sun, Moon, stars, planets and satellites was originally based on a model of circular uniform motion. Improved observational methods required an improved model which was found by superposition of different circles, each traversed by uniform motion. *Copernicus* seems to have been the first (or at least one of the first) to have recognized that the planets do not move in a circle. The new curve to be adapted to planetary motion was found by *Kepler*. Based on a general theory of adaptation of any motion as a consequence of the method of the variation of parameters, the present paper shows that the circle is the only curve which is not able to be adapted to any general motion.

Keywords: Circular motion, superposed circular motion, *Hansen Ideal* coordinates, *Hansen* system, *Leibniz* system, *Lagrange* constraint, method of variation of parameters, general *Frenet* formulae, general *Darboux* vector, system proper motion vector, Fourier expansion

Table of Symbols:

A_ν	Parameters of a curve presented in polar equation form (ν =1,2,3 …– number of parameter). In the unperturbed case: $A_\nu \equiv const$.
b_R, b_T, b_N	acceleration in radial, transversal, normal direction [km/s^2]
b_{RC}	acceleration [km/s^2] of a circular motion
b_{R0}	(radial) acceleration [km/s^2] without any physical influence
b_{RK}	Keplerian acceleration [km/s^2]
$\mathbf{D}_q$	system proper motion vector of $\mathbf{q}_j$ -system: $\mathbf{D}_q = D_q{}^j \mathbf{q}_j$
G	equal area parameter [km^2/s]
i	inclination [deg]
p	semilatus rectum [km]

$\mathbf{p}_i$	Basis vectors of an inertial system (*Newton* frame) (i=1,2,3)
$\mathbf{q}_j$	Basis vectors of an ideal system (*Hansen* frame) (j=1,2,3)
$\mathbf{R}$	inertial acceleration vector [km/s^2]
$\mathbf{r}, \dot{\mathbf{r}}$	state vector: direction vector, absolute velocity vector
$\mathbf{r}_0, \mathbf{q}_0, \mathbf{c}_0$	Basis vectors of co-moving orbit system (*Leibniz* frame), $\mathbf{r}_0$ in radial, $\mathbf{q}_0$ in transversal ("transradial"), $\mathbf{c}_0$ in normal direction
r	radius [km]
t	time [s]
u	argument of latitude [rad]
V	velocity [km/s]
$y_j, \dot{y}_j$	Cartesian coordinates in $\mathbf{q}_j$ -system
δ_{ij}	Kronecker tensor
ζ	orbit angle (first *Hansen* angle) [rad]
η	spatial rotation angle (second *Hansen* angle) [rad]
μ	centric gravitational constant [km^3/s^2]
σ	longitude in orbit of ascending node [rad] (related to departure point of *Hansen* system)
ξ	auxiliary angle [rad] for perturbational equations
Ω	right ascension (or ecliptical longitude) of ascending node [deg]

1 The Circle as a Model to describe Celestial Motions

The idea of the circular motion of the Sun, the Moon and the stars goes back to early Presocratian philosophers. *Thales of Miletus* (Θαλῆς, born about 640 BC) assumed a flat Earth disc floating on water and being totally surrounded by water. Therefore the Sun, the Moon and the stars move along an arc which is not specified in detail. However, based on daily observation with respect to the flat surface of the Earth, we may assume that this arc is not necessarily assumed to be a part of a circle, but rather a kind of a highly eccentric ellipse.

Anaximander of Miletus (Ἀναξίμανδρος, ca 610 - 545 BC), a younger contemporary of Thales[1], is referred to by many as the first philosopher and physicist. His contribution was a scientific revolution comparable to the *Copernican* revolution, possibly being of even greater importance. He seems to have been the first to imagine that the Earth, with the shape of a disc, is in suspension in the centre of the universe. Considering this, *Anaximander* extended the dimensions of the universe by assuming the diameter to be 56 Earth radii[2]. The primary matter

[1] J. L. E. Dreyer (1953), pp.11-16

[2] Other numbers as reported in J. L. E. Dreyer (1953) are not relevant for the topic of the present paper

is the "infinite" (ἄπειρον) which is divided into the four primary elements. These elements are in an eternal battle between the "terrestrial" elements, water and earth, and the "celestial" elements air, and around it, fire. As a consequence, there is a most extensive distance between these groups of elements. Therefore the primary position of fire is like a (circular) wheel at a great distance to the Earth. There are three wheels, one bearing the Sun which is shining through a small opening in the clouds hiding the rest of the wheel, a second wheel inside the Sun bearing the Moon, with a variable opening producing the lunar phases, and inside the Moon a third sphere with a lot of variable openings representing the stars (cf. Figure 1).

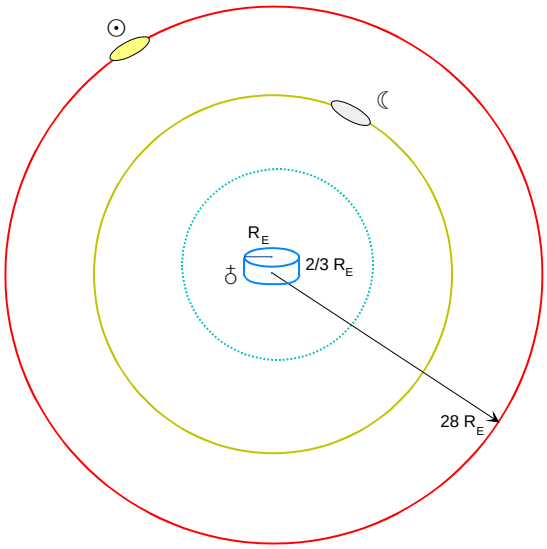

Figure 1: *Anaximander*'s view of the universe: the cylindrical Earth body (diameter three times the height) in free suspension, the sphere of the stars (inner circle), the wheel of the Moon (middle), the wheel of the Sun (outer circle). R_E = Radius of Earth = Radius of Earth disc

With this view of the universe, we can conclude that *Anaximander* introduced the circular motion of the Sun, the Moon and the stars as a physical appearance, and by no means as something that originates in a pseudo-religious superstructure and which would be found in myths.

A more precise modification of Anaximander's concept was prepared by *Pythagoras of Samos* **(**Πυθαγόρας, ca 571 - 497 BC). He assumed the Earth to represent a sphere, which was a very important step forward. He also seems to be the first to introduce the double motion astronomy: the daily motion of the stars with respect to the equator, the opposite motion of Sun and Moon with respect to the ecliptic. This is the first further step forward towards the variety of circular motions, the motions in different orbital planes inclined to each other. The deviating motions of the planets were not yet known at that time.

The idea of uniform and circular motion was picked up and continued by *Alkmaion of Kroton* (᾽Αλκμαίον, ca 500 BC) a physician who assumed that "all divine beings are always in motion: Moon, Sun, stars, the whole sky". This is cited by *Aristotle*[3]:

[3] De anima, A2,405a29pp.

κινεῖσθαι γὰρ τὰ θεῖα πάντα συνεχῶς ἀεί, σελήνην, ἥλιον, ἀστέρας καὶ τὸν οὐρανὸν ὅλον .

Parmenides of Elea (Παρμενίδης, ca 515 - 445 BC) assumed that there was a pure celestial air, called "ether", above which the perfect sphere encloses the finite universe. The stars are affixed to this sphere. However, *Parmenides* assumed that there was no motion of this sphere. There is, of course, no doubt about the circularity of the orbits of the Sun, the Moon and the stars. *Platon of Athens* (Πλάτων, 427 – 347 BC) deeply influenced by *Alkmaion* and *Parmenides*, stressed the term motion in all of his dialogues, especially in the dialogues *Timaios* and the laws (*Nomoi*). *Platon* assumed the motions of the celestial bodies to be of limited lifetime (in contrast to *Aristotle* who assumed eternal life for all these bodies). However, he assumed this motion as an image of divine harmony. In contrast to some contemporary assumptions, *Platon* required this motion to take place in only one orbit, which is a circle (κυκλοφορία):

> τὴν αὐτὴν γὰρ αὐτῶν ὁδὸν ἕκαστον καὶ οὐ πολλᾶς ἀλλὰ μίαν ἀεὶ κύκλῳ διεξέρχεται
> *„each of them uses the same orbit and not many but always a circle"* [4].

2 The Superposition of Circular Motions

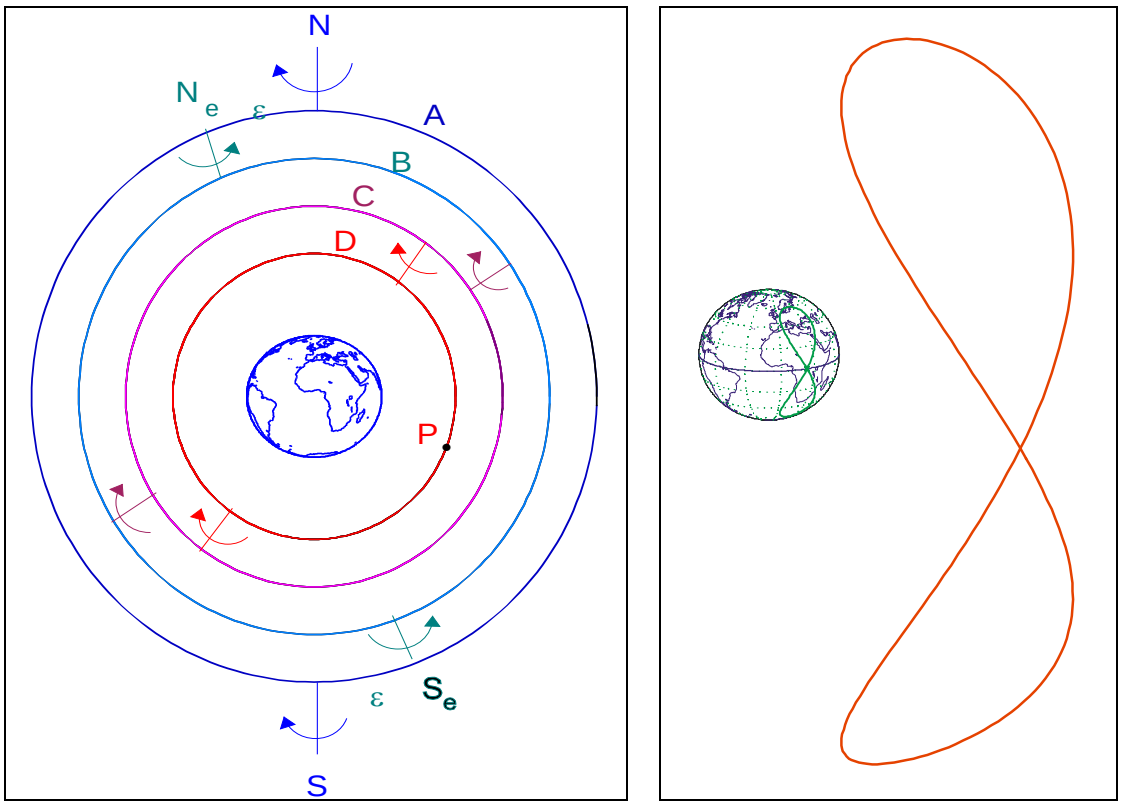

Figure 2: left figure: The system of the concentric spheres of *Eudoxos of Knidos* for the mathematical description of the motion of the planets[5]: A – Sphere of stars (equatorial system), *N* = equatorial North pole, *S* = equatorial South pole, B – Zodiac (Ecliptical system), ε = obliquity of the ecliptic, N_E = ecliptical North pole, S_E = ecliptical South pole, *C* and *D* - spheres to describe the second anomaly of the planetary motion, *P* = position of a planet (according to the assumed representation of the system of *Eudoxos*) Right figure: Hippopede (Lemniscate) of circular geosynchronous satellite orbits in true dimensions: 24^h orbit with inclination i = 60°.

[4] Nomoi 822a infr.

[5] Cf. J. L. E. DREYER (1953), p.90 and F. KRAFFT (1972), p.431

Only during the time of *Platon* was knowledge of the planets propagated in Greece. This new knowledge included the ”anomalous” motion of the planets showing loops and retrograde motions. Although *Platon* believed this behaviour to be fictitious, he asked the most eminent mathematician of this age, *Eudoxos of Knidos* (Εὐδόξως, ca 408 - 355 BC), to describe this motion by mathematical formulation. *Eudoxos* extended the well known double motion astronomy by two additional spheres describing the motion of each of the planets. By this combination of two circles on inclined orbital planes traversed with equal velocities, the relative motion with respect to the surface of the Earth will appear as a lemniscate, which was called a “Hippopede” in ancient times. This model still describes the unperturbed motion of a geosynchronous satellite moving on an inclined orbital plane at the same speed as the Earth revolves (cf. Figure 2). Here the well known “8” appears, a curve identical with the antique Hippopede. The anomaly described by the model of concentric spheres is known as the “second anomaly”. The “first anomaly”, reflecting the unequal length of the seasons and an eccentric motion of the Sun, was recognized long before *Platon*. *Euktemon* (ca 430 BC) observed the periods 93, 90, 90 and 92 days for spring, summer, autumn and winter, respectively. This anomaly was included in *Kallippos of Kyzikos'* (Κάλλιππος, ca 330 BC) model of concentric spheres with the assumption of two additional spheres to describe the solar motion. He also introduced another two additional spheres for the Moon. Therefore four different spheres, in addition to the spheres related to the equator and the ecliptic, were necessary to describe the motion of the celestial bodies. For the whole solar system he required 33 spheres.

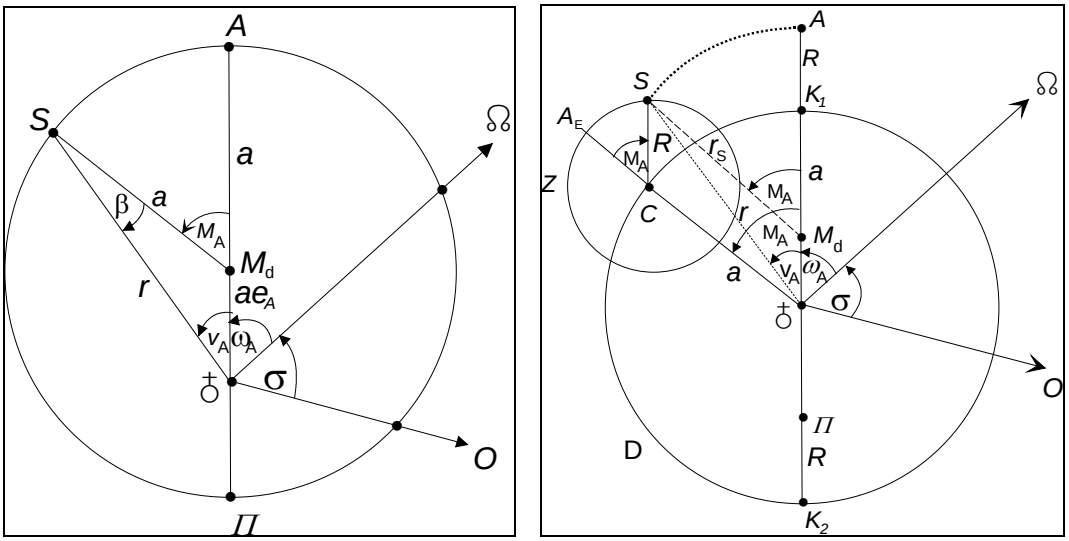

Figure 3: Left: Explanation of the eccentric circular motion, S –position of planet, A – apogee, Π - perigee, M_d - centre of circular motion, a – radius of eccentric circle, e_A – numerical eccentricity, υ_A – true eccentric anomaly (related to apogee A), M_A – mean eccentric anomaly, β - equation of centre (= Prostaphairesis), *O* – departure point, ☊ - ascending node of planetary orbit, σ- Longitude of ascending node in orbit, ω_A – argument of apogee, r – Earth-planet distance; Right: explanation of uniform epicyclic motion: R – radius of epicycle = $\overline{+\bigcirc M_d}$, D – Deferent (=main circle), Z – epicycle, C – centre of epicycle, A_E – apogee on epicycle

In contrast to this model, *Aristotle of Stagiros* (᾽Αριστοτέλης, 384 - 322 BC) introduced a physical model based on his fifth element, the celestial ether. Although *Aristotle* knew the difference between mathematical formulation and physical reality[6], he assumed that physical spheres carrying the celestial bodies have no proper motions apart from the motion of the spheres themselves. In order to stabilize this system, *Aristotle* had to introduce an additional system of "unrolling" spheres, so a total of 55 spheres became necessary to describe the motions of the celestial bodies. According to *Aristotle,* the first and only perfect, natural and infinite motion is circular motion[7]. All other motions, especially rectilinear motion, are imperfect.

The next important step towards the establishment of circular motion was the introduction of the eccentric circle model and epicyclic motion (see Figure 3), the brilliant mathematical formulation of which is due to *Apollonios of Perge* (᾽Απολλώνιος, ca 240 - 170 BC)[8]. The mathematical equivalence of both models, for the case of uniform motion, was proven later on by *Adrastas of Aphrodisias* (around the year 0)[9].

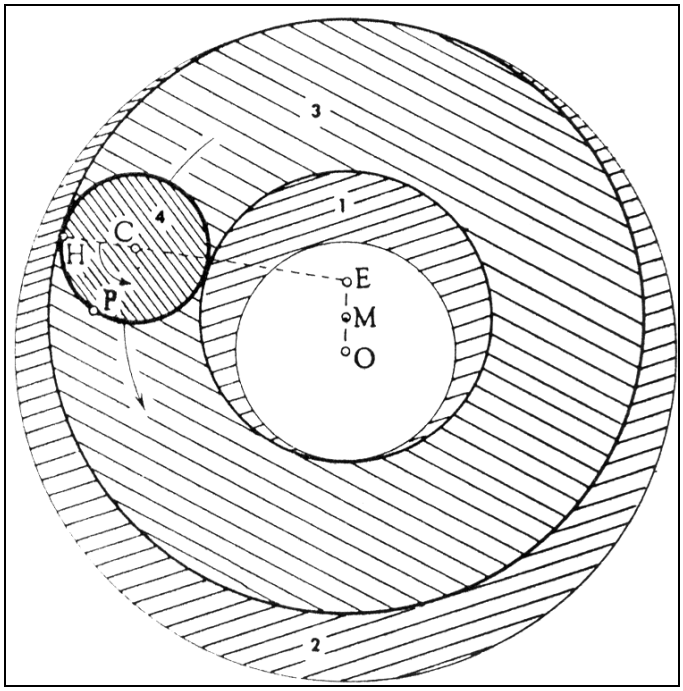

Figure 4: The mechanistic explanation of planetary motions by *Ptolemy* in the Planetary Hypothesis[10]. *O* – fixed point of observation (= Earth) within the centre of the inner empty circle (this is where the inner planets are located). *M* – centre of the spherical body 3, moving around *M* i , in such a way that the centre *C* of the spherical body 4 (the epicycle) would move uniformly with respect to the equant (punctum aequans) *E* . (However, with such an assumption the principle of uniform motion with respect to the axis of a spherical body would be violated). The epicycle will move uniformly with respect to its centre *C.* Fixed on this body, the planet *P* will move uniformly with respect to point *H*, showing an inclination (πρόσνευσις)□with respect to the equant *E* . The solid bodies, 1 and 2, complete the system so that in total the spherical arrangement of the whole system is retained with respect to the position *O* of the Earth.

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[6] As mentioned in book Z of the Metaphysics 1036 a pp. and in book M

[7] de caelo A II, 269 a

[8] O. NEUGEBAUER (1975), p. 262 pp.

[9] F. KRAFFT (1973), p. 61

[10] reproduction from: SWERDLOW, N.M and O. NEUGEBAUER (1984), p.615, Fig. 5

Ptolemy (Πτολεμαῖος, ca 100 - 178 AD) extended *Hipparchos'* (130 BC) lunar model by his discovery of the second inequality which, together with the first inequality (the eccentricity of the orbit), is nothing other than a combination of the equation of the centre and the evection. As a new mathematical model to describe this combined motion, *Ptolemy* used a combination of the eccentric and the epicyclic model. The second of his great discoveries was the introduction of the punctum aequans (the "equant") as the centre of equal motions: the line connecting the equant with the centre of the epicycle covers equal angles in equal times. Thus the combination of circular motions achieved a new and higher quality. Based on an improvement of the observations (partially prepared by *Ptolemy* himself), the mathematical description of the celestial motions became increasingly complicated. This was in contradiction to the assumption of *Platon* that the motion of the celestial bodies were to occur in a very simple manner. *Ptolemy* accepted this, thinking that only the mathematical description was complicated, not the physical reality. He wrote[11]: "*It is necessary to always adapt the simplest hypothesis to the motions in the sky. Only if this way is unsuccessful then a more complicated but adaptable hypothesis should be used*". The Almagest, the textbook in Astronomy for about 1500 years, was the first of the great works published by *Ptolemy*. In the course of his life he tried to improve his ideas and published the improvements in astronomy (among them an improved theory of the planetary latitudes) in his last work, the "Planetary Hypothesis". Parts of this work survived but only in its Arabic translation (known by the name $kit\bar{a}b\,al-manshur\bar{a}t$) and was not rediscovered in the west until 1967[12]. In this book, *Ptolemy* tried to prepare a mechanical model to describe the planetary motions. He assumed that a planetarium could be constructed based on this model (see Figure 4).

3 The End of the Idea of Circular Motion - the End of the Antique Theory of Planetary Motion

There was only little progress in mathematical modelling of orbital motion in the period after *Ptolemy*. *Proclus* (AD 412 - 485) derived a mathematical combination of circular and rectilinear motion, which was extended by $Nasir\,ad-Din\,at-Tusi$ (died AD 1274), the founder of the observatory in $Mar\bar{a}gha$ in 1259. $At-Tusi$'s device improved the theory of planetary latitudes using additional circular motions. He and his students criticized *Ptolemy*'s physical model with its non-uniform motion of the planets with respect to the Earth and tried to improve the model of planetary motion by introduction of additional epicycles. In this manner, $Ibn\,ash-Sh\bar{a}tir$ (AD 1304 – 1375/76) in Damascus found a model for lunar motion containing a first epicycle representing the equation of the centre and a superposed second epicycle for the evection[13]. This model, describing lunar motion as a geocentric motion, could be adapted by *Copernicus* in "*de revolutionibus*" without any modification. The model of planetary motion by $Mulaqqad\,ad-Din\,al-'Urdi$ (died AD 1266) and

[11] PTOLEMAIOS, Mathematike Syntaxis XIII 2, Teil 2, p.532; included in: Claudii Ptolemaei opera quae exstant omnia, edidit by J. L. HEIBERG, Leipzig 1898-1903; cited in: F. KRAFFT (1973), p.68

[12] GOLDSTEIN, B. (1967)

[13] ROBERTS, V. (1957)

Qutb ad – Dīn ash – Shīrāzī (AD 1236-1311) used two epicycles of totally different sizes to describe the motion of the planets. This model was transposed by *Copernicus* to his heliocentric model, erasing one of the epicycles representing the shift from the Earth as the centre, to the Sun as the centre of the motion.

The resulting model was drawn by *Copernicus* himself in his manuscript *de revolutionibus*, as shown in Figure 5.

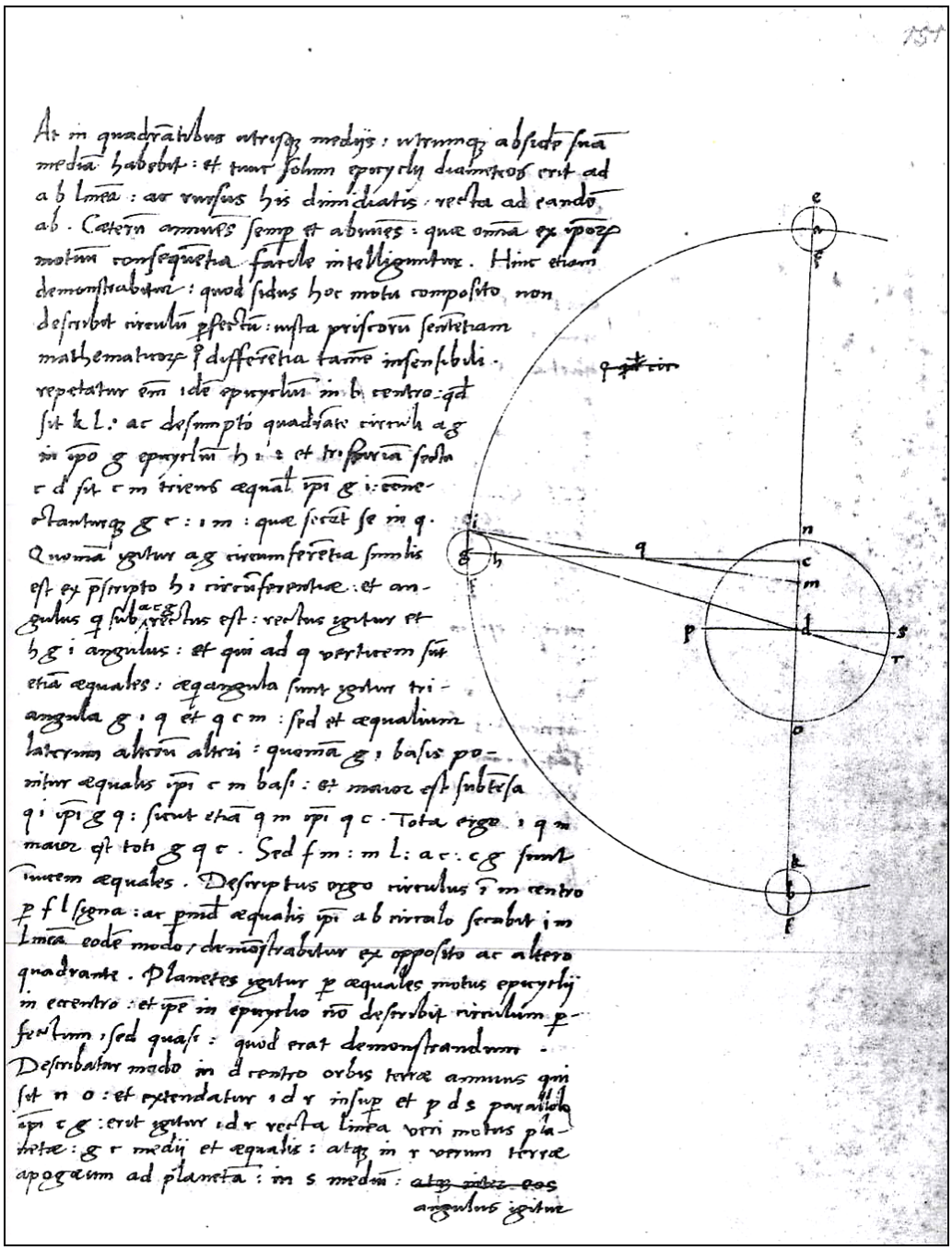

Figure 5: Copy of the facsimile issue of the handwritten original of de revolutionibus (1944), this page including the reference to the non-circularity of the planetary motions and the theory of the (outer) planets based on the transposition of the planetary model of *al–'Urdī* and *ash – Shīrāzī*

The corresponding mathematics is summarized in Figure 6. It shows the coordinates of a planet as presented in an Aphel oriented, planar orbit related, rectangular coordinate system

$$\begin{aligned} x_A &= r\cos\upsilon_A = \frac{3}{2}a\,e \;+\; a\cos M_A \;-\; \frac{1}{2}a\,e\cos 2M_A \\ y_A &= r\sin\upsilon_A = \qquad\quad a\sin M_A \;-\; \frac{1}{2}a\,e\sin 2M_A \qquad . \end{aligned} \tag{1}$$

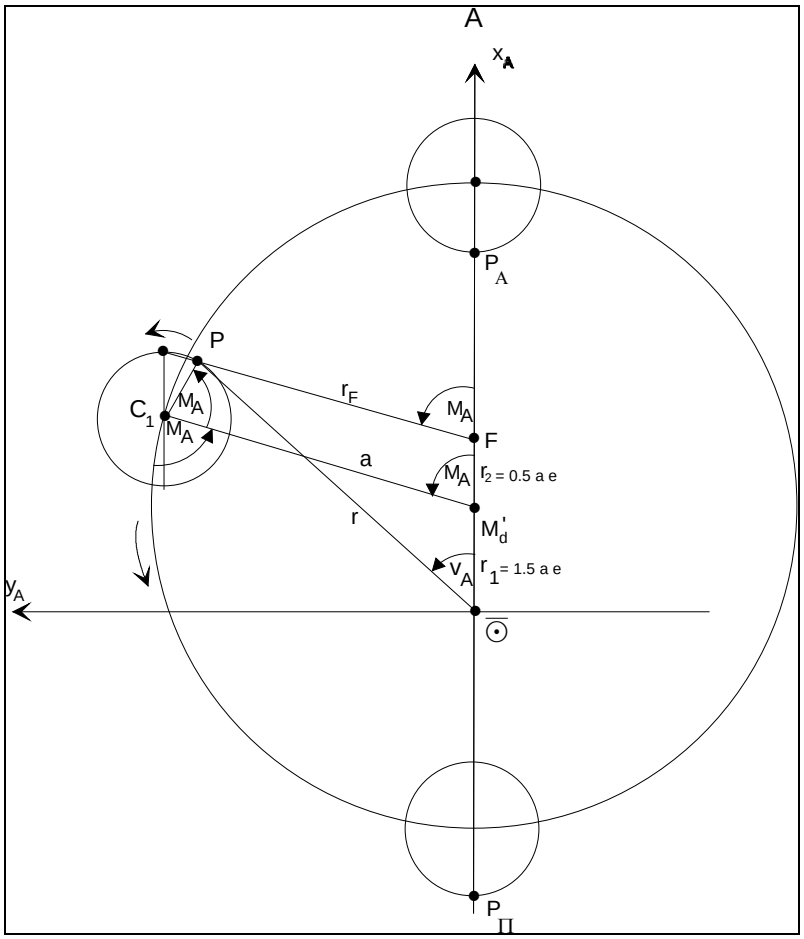

Figure 6: The Copernican theory of the motion of the outer planets in their orbital plane: x_A, y_A – coordinate system of the orbital plane related to the Aphel P_A, P_Π- Perihel, υ_A = true eccentric anomaly, M_A – mean eccentric anomaly, M'_d - centre of the deferent according to *al–'Urdī* and *ash – Shīrāzī* /*Copernicus*, *F* – punctum aequans; motion related to the mean Sun

This is nothing but a *Fourier* expansion[14] of elliptic motion up to the order of accuracy known to *Copernicus* (about 10 arc minutes or even worse[15]), therefore a mathematical (and not a

[14] KAMLAH, A. (1971); perhaps detected for the first time by G.V. Schiaparelli (see BOCCALETTI, D. AND PUCACCO, G. (1996), p.4)

[15] Cf. V. BIALAS (1972) and private communication Nov. 2005: the error of Copernicus's own measurements was in the order of 20 arcminutes, composed by uncertainties in the places of the stars, measurement errors and errors in the mean places of the Sun

hypothetical) model of planetary motion based on perfect relevance. The remark of *Copernicus* himself (as shown in Figure 5) seems to be of extreme importance:

"*Hinc etiam demonstrabitur, quod sidus hoc motu compositu, non describit circulum perfectum iuxta priscorum sententia Mathematicorum, differentia insensibili"*

("*in consequence of this composed motion, the star does not describe a perfect circle in accordance to the meaning of previous mathematicians, however, the deviation is negligible*")[16].

This sentence includes three interesting statements which must be accepted as recognitions of *Copernicus* himself:

1. A qualitative statement: the planets do not move in circles. This is the first time in history that, based on mathematical issues, the non-circularity of the orbits of the celestial bodies was accepted. This is also an absolute statement: with mathematical relevance a curve can be a circle or it cannot be a circle (tertium non datur), even if this curve can be represented by superposition of different circles.
2. A quantitative statement: the deviation of the true orbit of the planet's motion with respect to a circular curve is negligible. Lacking accurate measurements, *Copernicus* had probably no insight into the importance of this statement. It was only three years after *Copernicus*' death that an astronomer was born who was able to confirm the considerations of *Copernicus* by means of the incredible accuracy of his naked eye observations of 2 arc minutes and better. Based on these observations, the *Fourier* expansion (1) had to be extended up to the second order in eccentricity. However, this expansion shows that no more superposed circles could describe the planetary motion.

   The author would like to emphasize that the qualitative and the quantitative statements must be regarded independently.
3. A historical statement concerning references to prior mathematicians: *Copernicus* did not mention any names or origins of these persons. However, we should accept that these people existed and we are quite sure that only (or especially, if among others) the astronomers of Marāgha are meant[17].

Based on *Tycho Brahe*'s (AD 1546 - 1601) excellent observations of the planetary motions, especially of the planet Mars, *Johannes Kepler* (1571 – 1630) finally stated that no circular motion and also no longer any superposition of circular motions could perfectly describe planetary motion. *Kepler* perceived the ellipse as the curve relevant to all celestial bodies and assumed to have found a general law. However, *Kepler* did not respect precession or any perturbations of spherical astronomy, in particular he did not include any mutual influence between the bodies in the solar system, although he supposed a $1/r^2$ law of force existed between the bodies. *I. Newton* (1643-1727), who addressed the physical interpretation of the Keplerian motion, was the first to consider the mutual attraction between all celestial bodies based on his law of gravitation. However, only *L. Euler* (1707-1783) succeeded in establishing a mathematical means of solving any problem of motion by his method of the variation of parameters. The basic curve to adapt to any motion is usually a conic section, especially the ellipse in case

[16] de revolutionibus 142V10-12. See also SWERDLOW, N. M. und O. NEUGEBAUER (1984), p. 296, n.17.

[17] Some scientists will not „believe" that these astronomers exist. Why should we not believe to that what *Copernicus* himself states?

of periodic motions. In 1762, *Euler* published for the first time the perturbations in elliptic elements, by his method of successive approximation, in his paper *„Nouvelle méthode de déterminer les dérangemens dans le mouvement des corps célestes, causés par leur action mutuelle“*[18]. He described the method of the variation of the parameters in his paper *„Considérations sur le Problème des Trois Corps“*[19], where he also mentioned the orbital elements known today as “Keplerian elements”. It was *J. L. Lagrange* who derived the general variational equations for the Keplerian elements in case of conservative accelerations, whereas *C. F. Gauß* extended these equations for the case of non-gravitational accelerations. An important tool to solve these equations was presented by the method of canonical variables by *W. R. Hamilton*, in the version of *H. von Zeipel*, and the application by *D. Brouwer* to solve the main problem of satellite theory.

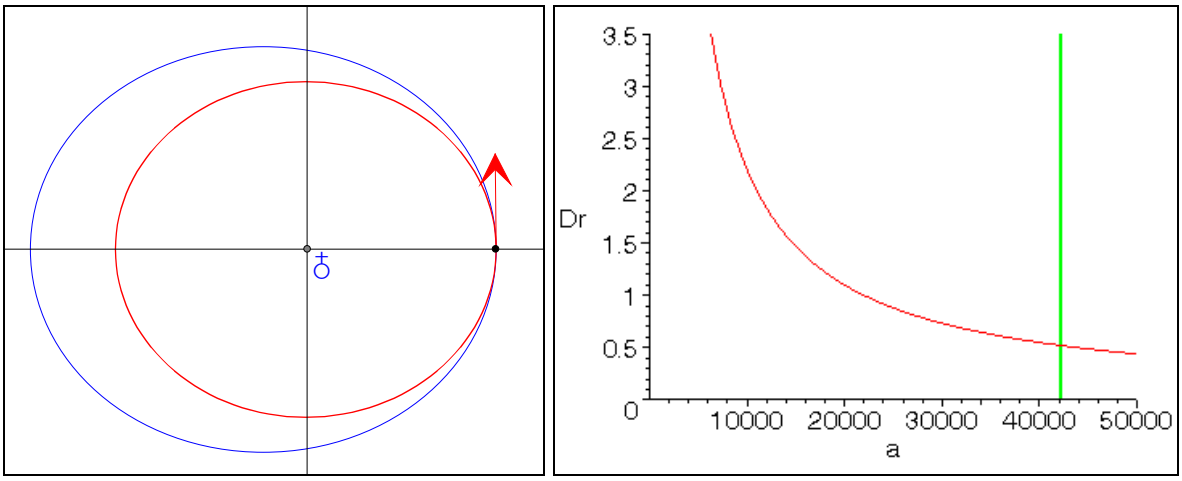

Figure 7: The motion of an equatorial circular body under the influence of J_2 only in the pericentre of its osculating ellipse. The right plot shows the values in case of Earth satellites, deviation Δr[km] versus semi-major axis a[km]. Geostationary satellites: a_0 = 42164.182 km, Δr = 0.52226 km, oscul. semi-major axis: a = 42166.271 km

In the context of circular motions, a surprising discovery was made by *R. H. Lyddane* (1963). He was the first to transfer the *Brouwer* solution from *Delaunay* elements to a new set of non-singular parameters. In this context, he could show that under the influence of the main zonal effect J_2 of a planetocentric gravitational field, a circular orbit of a satellite would be possible only if the satellite were always to be in the pericentre of its orbit, i.e. the argument of pericentre and the true anomaly of the orbit are identical (apart from a constant factor) (cf. Figure 7). Let a_0, e_0, i_0 = 0°.0, the *Kepler*ian elements of an unperturbed equatorial orbit, then the osculating radius r of the satellite under the influence of J_2 , the semimajor axis a, and the eccentricity e will be[20]

$$r = r_0 + \Delta r = r_0 + \frac{1}{2} J_2 \frac{R_E^{\,2}}{a_0} + O\left(a_0 B_2^2\right), \; a = a_0 + \frac{1}{4}\Delta r \;, \; e = \frac{3}{2} J_2 \left(\frac{R_E}{a_0}\right)^2 + O\left(a_0 B_2^2\right) \;, \quad (2)$$

[18] L. EULER collected works, E.398; 8. July 1762

[19] L. EULER , Mémoires acad. roy. sci. et belles-lettres **19**, 1770, pp.194-220

[20] JOCHIM, E. F. AND ECKSTEIN, M. C. (1980)

where R_E is the mean equatorial radius of the main body and

$$a_0 = r_0 \ , \ B_2 = \frac{3}{2} J_2 \left(\frac{R_E}{a_0} \right)^2 \quad . \tag{3}$$

Let ω be the argument of pericentre, Ω the longitude (or right ascension) of the ascending node, v the true anomaly, L=v+ω+Ω the true longitude then

$$\frac{dL}{dt} - \frac{d\left(\omega + \Omega\right)}{dt} = O\left(B_2^2\right) \tag{4}$$

and, in consequence, the mobile body (Moon, Planet, satellite, orbiter) is moving with the ("unperturbed") *Kepler*ian mean motion

$$n_0 = \sqrt{\frac{\mu}{a_0^3}} \tag{5}$$

staying always in the pericentre of the osculating ellipse with form parameters a and e. For inclined orbits, no circular motion is possible under these conditions. With respect to higher order physical influences, the true orbit will always deviate from circular motion.

4 On a General Theory of Adaptation of any Motion

Extending the *Euler*ian method of the variation of parameters, we come to a more general understanding of the adaptation of any motion, as was first proposed by the antique astronomers and more precisely formulated by *Ptolemy*. Let **r** be the radius vector of any moving body with respect to a reference point (which is not necessarily a centre of gravitation) and let $\mathbf{r}_0, \mathbf{q}_0, \mathbf{c}_0$ be the direction vectors of the widely used accompanying trihedron, as was first introduced by *G. W. Leibniz* (1689) into celestial mechanics[21], then the variation with respect to time t leads to

$$\mathbf{r} = r\ \mathbf{r}_0 = \left|\mathbf{r}\right| \mathbf{r}_0 \quad , \quad \dot{\mathbf{r}} = \dot{r}\,\mathbf{r}_0 + r\ \dot{\mathbf{r}}_0 \quad . \tag{6}$$

Let the value of the variation of the radial unity vector $\mathbf{r}_0$ be

$$\dot{\zeta} := \left|\dot{\mathbf{r}}_0\right| \quad , \ \dot{\mathbf{r}}_0 = \dot{\zeta}\ \mathbf{q}_0 \quad , \tag{7}$$

then it can be shown[22], that the integral ζ will be the one and only angle of the celestial body related to a *Hansen* coordinate system. *P. A. Hansen* (1857 and in previous papers) established a coordinate system in which "perturbed and unperturbed formulae have identical form". *Hansen* called such a system an "ideal" system. In a similar but more generalized sense, he uses the term "ideal" for a behaviour adapting the same form in the perturbed, as well as in the unperturbed, case.

The equations (6) and (7) lead to the orbit normal vector[23]

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[21] cf. AITON, E. J. (1960), this system will sometimes be called „orbit system" in satellite orbital mechanics

[22] JOCHIM, E. F. M. (2011/2), theorem 10 and theorem 12

[23] The value of the orbit normal vector will be designed with the symbol *G* according to the corresponding canonical element in *Ch. Delaunay*'s theory and is in this form already widely used in Astrodynamics

$$\mathbf{c} \;=\; \mathbf{r}\times\dot{\mathbf{r}} \;=\; r^2\,\dot{\zeta}\,\mathbf{c}_0 \;=\; G\,\mathbf{c}_0 \quad , \quad \left(\mathbf{c}_0{}^2 = 1\right) \quad . \tag{8}$$

This expression, especially including the general relation

$$r^2\,\dot{\zeta} \;=\; G \;, \tag{9}$$

is absolutely independent of any orbital form and corresponds to the initially detected Second *Kepler*ian Law.

Let $\mathbf{q}_j$ be the basic unit vectors of a *Cartesian* orbital, plane-related, coordinate system, where $\mathbf{q}_1$ is oriented towards a departure point O on the orbit with angular distance ζ to the moving body. Then

$$\mathbf{r} \;=\; y^j\,\mathbf{q}_j \;=\; r\left(\mathbf{q}_1\cos\zeta + \mathbf{q}_2\sin\zeta\right) \;=\; r\,\mathbf{r}_0 \tag{10}$$

and

$$\dot{\mathbf{r}} \;=\; \dot{y}^j\,\mathbf{q}_j + y^j\,\dot{\mathbf{q}}_j \quad . \tag{11}$$

Applying (7), we have

$$y^j\,\dot{\mathbf{q}}_j \;=\; 0 \quad , \tag{12}$$

i.e. the condition for a *Hansen* system is fulfilled. This system is unvaryingly bound to the (osculating) orbital plane, therefore it is possible to set $y_3 = 0$, so equation (12) leads to

$$\dot{\mathbf{q}}_1\cos\zeta \;+\; \dot{\mathbf{q}}_2\sin\zeta \;=\; 0 \quad . \tag{13}$$

The system variation vectors $\dot{\mathbf{q}}_j$ are usually represented by

$$\dot{\mathbf{q}}_j \;=\; \mathbf{D}_q\times\mathbf{q}_j \tag{14}$$

where $\mathbf{D}_q$ is the system proper motion vector (sometimes misleadingly denoted as “rotational vector”, however, in comparison to the *Darboux* vector of curvature theory, this vector could also be called “system *Darboux* vector” or “general *Darboux* vector” or simply “*Darboux* vector”). From equations (10) and (13), follow the behaviour of a *Hansen* system[24]

$$\mathbf{D}_q\times\mathbf{r} = 0 \tag{15}$$

The consequence is very important: if an integration of the equations of motion takes place in a *Hansen* system, no proper motion of this system has to be respected because this system is always connected with the motion of the celestial body. An historical dispute between *G. W. Leibniz* and *I. Newton* is reflected in this fact: an integration of the equations of motion always has to be executed in a coordinate system whose proper motion vanishes. In his Principia, *I. Newton* only uses “inertial coordinates”[25], i.e. coordinates whose proper motion will be neglected. Today, such a system is defined, for example, by the Fundamental Catalogue FK5 and is related to the mean equinox J2000.0. In this context, the equation of motion is

$$\ddot{\mathbf{r}} \;=\; \mathbf{R} \;=\; a^i\,\mathbf{p}_i \tag{16}$$

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[24] as found by *P. A. Hansen* himself in 1857

[25] See e.g. in AITON, E. J. (1964): p.32

where **R** is the (inertial) acceleration vector and a^i its components with respect to the inertial (="Fundamental system") $\mathbf{p}_i$ (i=1,2,3)[26]. The correct way to undertake integrations of the motion of planets, and artificial satellites or space probes, is to transform the actual orbital data into this system, perform the integration within this system and afterwards transform the actual data back to the true system of date. Based on *Leibniz*' co-moving coordinate system with accelerations b_R, b_T, b_N, as used by *C. F. Gauß* in his perturbation theory, the equation of motion holds

$$\ddot{\mathbf{r}} \;=\; b_R\,\mathbf{r}_0 \;+\; b_T\,\mathbf{q}_0 \;+\; b_N\,\mathbf{c}_0 \quad . \tag{17}$$

From equations (6), it follows

$$\ddot{\mathbf{r}} \;=\; \ddot{r}\,\mathbf{r}_0 + 2\,\dot{r}\,\dot{\zeta}\,\mathbf{q}_0 + r\,\ddot{\zeta}\,\mathbf{q}_0 + r\,\dot{\zeta}\,\dot{\mathbf{q}}_0 \quad . \tag{18}$$

Introducing the parameters $\dot{\xi}, \dot{\eta}$, and because $\mathbf{q}_0{}^2 = 1$ only in the plane perpendicular to $\mathbf{q}_0$, the variational vector

$$\dot{\mathbf{q}}_0 \;=:\; \dot{\xi}\,\mathbf{r}_0 + \dot{\eta}\,\mathbf{c}_0 \tag{19}$$

will be defined and the acceleration vector becomes

$$\ddot{\mathbf{r}} \;=\; \left(\ddot{r} + r\,\dot{\zeta}\,\dot{\xi}\right)\mathbf{r}_0 + \left(2\,\dot{r}\,\dot{\zeta} + r\,\ddot{\zeta}\right)\mathbf{q}_0 + r\,\dot{\zeta}\,\dot{\eta}\,\mathbf{c}_0 \quad . \tag{20}$$

Therefore, the transversal acceleration leads to the variational equation for the parameter G based on the transversal acceleration only

$$\dot{G} \;=\; r\;b_T \tag{21}$$

and the variational equation for the parameter η based on the normal acceleration only

$$\dot{\eta} \;=\; \frac{r}{G}\,b_N \quad . \tag{22}$$

According to equation (10), the transversal direction vector in the *Hansen* coordinate system is written as

$$\mathbf{q}_0 \;=\; -\,\mathbf{q}_1 \sin\zeta + \mathbf{q}_2 \cos\zeta \tag{23}$$

and the variation

$$\dot{\mathbf{q}}_0 \;=\; -\,\dot{\zeta}\left(\mathbf{q}_1 \cos\zeta + \mathbf{q}_2 \sin\zeta\right) - \dot{\mathbf{q}}_1 \sin\zeta + \dot{\mathbf{q}}_2 \cos\zeta \quad . \tag{24}$$

By applying the *Hansen* property (13), it is known that $\dot{\mathbf{q}}_1$ and $\dot{\mathbf{q}}_2$ are oriented in the identical direction and perpendicular to the plane formed by the basic vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. Therefore, $\dot{\mathbf{q}}_1$ and $\dot{\mathbf{q}}_2$ have the same direction as $\mathbf{q}_1 \times \mathbf{q}_2 = \mathbf{q}_3 = \mathbf{c}_0$. It follows $\dot{\mathbf{q}}_1 = \left|\dot{\mathbf{q}}_1\right|\mathbf{c}_0$, $\dot{\mathbf{q}}_2 = \left|\dot{\mathbf{q}}_2\right|\mathbf{c}_0$ and, including the radial direction vector (10),

$$\dot{\mathbf{q}}_0 \;=\; -\,\dot{\zeta}\,\mathbf{r}_0 - \left(\left|\dot{\mathbf{q}}_1\right|\sin\zeta - \left|\dot{\mathbf{q}}_2\right|\cos\zeta\right)\mathbf{c}_0 \quad . \tag{25}$$

Comparison with the variation of the transversal direction vector in equation (19) leads to

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[26] in *Newton*'s (et al.) time, the terms force and acceleration were mixed, disregarding *Newton*'s Second Law of Motion. However, in the era of space flight this should be avoided, also *Newton*'s Law of Gravitation should not be used as a basis for the equations of motion of a celestial body, e.g. an Earth satellite.

$$\begin{aligned} \dot{\xi} &= -\dot{\zeta} \\ \dot{\eta}\,\mathbf{c}_0 &= -\dot{\mathbf{q}}_1 \sin\zeta + \dot{\mathbf{q}}_2 \cos\zeta \quad . \end{aligned} \tag{26}$$

Therefore, equations (17) and (20) lead to the radial acceleration

$$b_R = \ddot{r} - r\dot{\zeta}^2 \tag{27}$$

or the radial acceleration of the moving body

$$\ddot{r} = \frac{G^2}{r^3} + b_R \quad . \tag{28}$$

This equation was found by *G.W. Leibniz* (1689) in the special case of the *Kepler*ian acceleration[27]

$$b_R \triangleq b_{RK} = -\frac{\mu}{r^2} \tag{29}$$

based on the 2-body problem

$$\ddot{r} = \frac{G^2}{r^3} - \frac{G^2}{pr^2} = \frac{G^2}{r^3} - \frac{\mu}{r^2} \quad , \tag{30}$$

where the first notation was derived by *Leibniz* using the first two *Kepler*ian Laws only, whereas the second notation, including the central gravitational constant μ, includes the 3rd *Kepler*ian Law. Similarly, but independently, *I. Newton* derived his Law of Gravitation on the basis of the *Kepler*ian Acceleration.

By applying the equations (7) and (19) and the relation $\mathbf{c}_0 = \mathbf{r}_0 \times \mathbf{q}_0$, we have $\dot{\mathbf{c}}_0 = -\dot{\eta}\mathbf{q}_0$. Therefore. the trihedron $\mathbf{r}_0$, $\mathbf{q}_0$, $\mathbf{c}_0$ will satisfy the system

$$\begin{aligned} \dot{\mathbf{r}}_0 &= \dot{\zeta}\,\mathbf{q}_0 \\ \dot{\mathbf{q}}_0 &= -\dot{\zeta}\,\mathbf{r}_0 + \dot{\eta}\,\mathbf{c}_0 \\ \dot{\mathbf{c}}_0 &= -\dot{\eta}\,\mathbf{q}_0 \quad . \end{aligned} \tag{31}$$

The structure of these equations is comparable to the *Frenet* formulae of the curvature theory. These formulae similarly represent a co-moving trihedron and could therefore be called "*Frenet* formulae of orbital mechanics" (or, particularly, "of Astrodynamics").

The third equation of system (31) prompts a geometrical interpretation of the parameter $\dot{\eta}$ as *R. A. Struble* performed in 1960: it describes the angular velocity of the rotation of the orbital plane with respect to the instantaneous radius vector (cf. Figure 8). The rotational angle η will be obtained by integration. In fact it is "invisible" because it only becomes visible by its variation. The angle η could be called the „second *Hansen* angle" if one considers the orbit angle ζ to be called the "first *Hansen* angle". However, the variational equation (22) shows a dependency of η upon ζ, for which can be derived by applying (9)

[27] JOH. KEPLER (1953), Epitome IV, 2, KGW Bd. VII, p.304, also: letter 1. Okt. 1602, KGW XIV,280, line 653-657; see also in HOYER (1976), p.199, U. HOYER (1979), p.71, V. BIALAS (2004) p. 91

$$\frac{d\eta}{d\zeta} = \frac{r^3}{G^2} b_N \quad . \tag{32}$$

The variation of the rotational angle η is connected with the inclination *i*, the right ascension of the ascending node Ω, and the longitude σ in the orbital plane by the well-known variational equations related to the inertial fundamental system

$$\begin{aligned} (i)^{\cdot} &= \dot{\eta}\cos u \\ \dot{\Omega}\sin i &= \dot{\eta}\sin u \qquad (\dot{\mathbf{p}}_i \equiv 0) \\ \dot{\Omega}\cos i &= \dot{\sigma} \quad . \end{aligned} \tag{33}$$

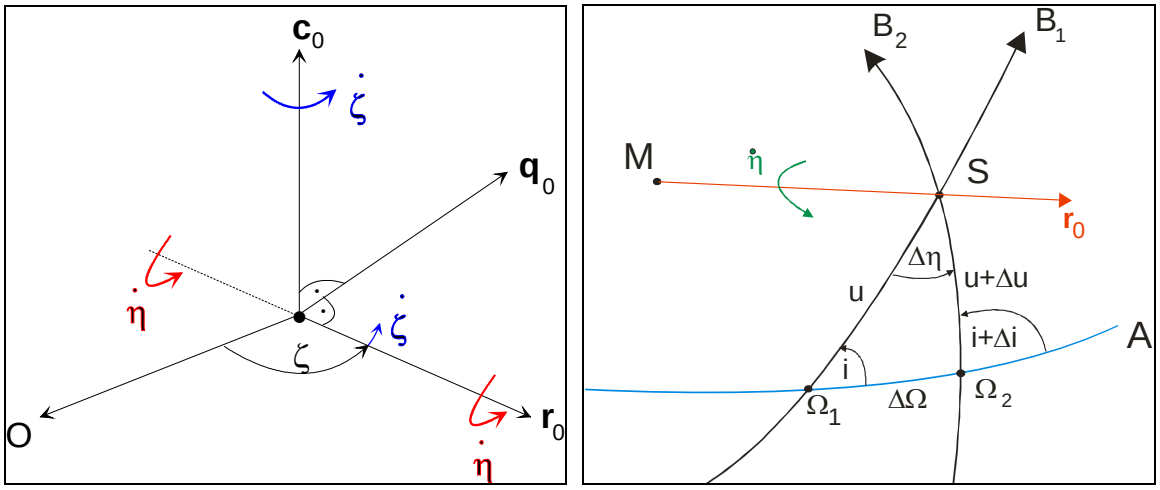

Figure 8, Left: rotation of the orbital plane around the radius vector with angular velocity $\dot{\eta}$. Right:: detection of the "invisible" angle η in case of normal acceleration of the celestial body at position S

Here $u = \zeta - \sigma$ is the argument of latitude as the node related part of the orbital angle. The angle η can be visualized by applying the normal acceleration b_N related to an uninfluenced ("unperturbed") orbital plane with inclination $i^{(0)}$, node $\Omega^{(0)}$ and longitude $\sigma^{(0)}$ of the node. A rotation to the new position of the plane with inclination $i^{(1)}$, node $\Omega^{(1)}$ and longitude $\sigma^{(1)}$ will be according to the spherical triangle $\Omega^{(0)}$, $\Omega^{(1)}$, *S* with the equations

$$\begin{aligned} \cos\left(\zeta - \sigma^{(1)}\right)\sin i^{(1)} &= \cos\left(\zeta - \sigma^{(0)}\right)\sin i^{(0)}\cos\eta + \cos i^{(0)}\sin\eta \\ \sin\left(\zeta - \sigma^{(1)}\right)\sin i^{(1)} &= \sin\left(\zeta - \sigma^{(0)}\right)\sin i^{(0)} \\ \cos i^{(1)} &= -\cos\left(\zeta - \sigma^{(0)}\right)\sin i^{(0)}\sin\eta + \cos i^{(0)}\cos\eta \quad . \end{aligned} \tag{34}$$

Here the parameters $i^{(1)}$, $\sigma^{(1)}$ are dependent parameters whereas $i^{(0)}$, $\sigma^{(0)}$ (or $\Omega^{(0)}$) are initial constants related to the uninfluenced orbit. This way, two integrational constants are fixed, the third will be obtained by the integration of equation (21) with respect to the orbital angle ζ

$$\frac{dG}{d\zeta} = \frac{r^3}{G} b_T \quad , \tag{35}$$

The fourth constant will be obtained from the integration of equation (9) in order to get the position of the celestial body against time

$$t = t_0 + \int_{\zeta_0}^{\zeta} \frac{G}{r^2} d\zeta \quad . \tag{36}$$

The remaining two integrational constants will characterize the curve describing the motion of the celestial body. The polar equation to describe this curve is given by

$$r = r(\zeta) = r(A_\nu; \zeta) \quad , \tag{37}$$

where A_ν (ν = 1,2,3, …) are the parameters of this curve. Only two independent parameters are normally needed. With this special curve the real motion should be adopted. This will be treated by selection of a variation of the parameters A_ν according to the method of variation of the parameters. If the radial velocity,

$$\dot{r} = \frac{dr}{dt} = \frac{\partial r}{\partial t} + \sum_\nu \frac{\partial r}{\partial A_\nu} \frac{dA_\nu}{dt} \tag{38}$$

then for the uninfluenced ("unperturbed") and the influenced ("perturbed") motion to coincide, the necessary condition will be

$$\frac{dr}{dt} = \frac{\partial r}{\partial t} \quad . \tag{39}$$

According to *P. A. Hansen* (1857), this means that the parameters for the adaptation of any motion must be "ideal" parameters. This shows the theory of *Hansen* in a very interesting new light. Consequently the *Lagrange* constraint[28] holds:

$$\sum_\nu \frac{\partial r}{\partial A_\nu} \frac{dA_\nu}{dt} = 0 \quad \text{or} \quad \sum_\nu \frac{\partial r}{\partial A_\nu} \frac{dA_\nu}{d\zeta} = 0 \quad . \tag{40}$$

If, for example, only two independent parameters have to be considered, then

$$\frac{\partial r}{\partial A_1} \frac{dA_1}{d\zeta} + \frac{\partial r}{\partial A_2} \frac{dA_2}{d\zeta} = 0 \quad . \tag{41}$$

This is the first equation to compute the parameters A_ν as a function of the orbital angle ζ (or the time resp.). The second equation will be the (general) *Leibniz* equation (28)

$$\frac{d\dot{r}}{d\zeta} = \frac{\partial \dot{r}}{\partial \zeta} + \sum_{\nu=1}^{2} \frac{\partial \dot{r}}{\partial A_\nu} \frac{dA_\nu}{d\zeta} = \frac{G}{r} + \frac{r^2}{G} b_R \quad . \tag{42}$$

Here, in any case it can be assumed

$$\frac{\partial \dot{r}}{\partial \zeta} = \frac{G}{r} \quad . \tag{43}$$

It remains for the second equation of adaptation

$$\sum_{\nu=1}^{2} \frac{\partial \dot{r}}{\partial A_\nu} \frac{dA_\nu}{d\zeta} = \frac{r^2}{G} b_R \quad . \tag{44}$$

[28] See e.g. EFROIMSKY, M. (2005)

Integration of any orbit using the variational equations above is treated with respect to the *Hansen* system. In this system the coordinate system proper motion vanishes as in the inertial fundamental system. It is also interesting to see another surprising consequence of the *Hansen* theory which is that both kinds of orbit integrations take place in a coordinate system with similar properties, even though the basic theoretical structure of these systems is fundamentally different.

5 The Inability of the Circle to Adapt any Motion

A circle is a curve with the radius as its only parameter. The radius will not vary during a motion on the circle. If $A_1 = r$ = const. and $A_2 = 0$ will be set, equation (41) gives

$$\frac{\partial r}{\partial A_1}\frac{dA_1}{d\zeta} = \frac{dA_1}{d\zeta} = 0$$

and again, by integration, $A_1 = r$ = const. . The *Leibniz* equation (28) leads to the acceleration of a circular motion $b_{RC} = -G^2/r^3$ whose value is identical to the condition for a uniform rectilinear motion, and so for a circular motion it is $\ddot{r} = 0$. With mathematical methods there is no chance to escape from the circle. However an escape is possible by physical means. Let b_R be any radial acceleration. Related to a circular motion with acceleration $b_{RC,}$ the acceleration "perturbing" the circular motion will be

$$b_{R0} = b_R - b_{RC} = b_R + \frac{G^2}{r^3} \quad .$$

When introduced into the general *Leibniz* equation (28), the radial acceleration will be

$$\ddot{r} = b_{R0} \quad ,$$

which ist the acceleration of a motion without **any** pre-adapted curve. An adaptation based on a circle is not recognizable.

6 Conclusion

The theory of the general adaptation of arbitrary motions, which is deducible from *Hansen*'s coordinate system, has as a necessary consequence the *Euler*ian method of variation of parameters in the version of the *Lagrange* constraint. The unique behaviour of the circle to be unable to be adapted to any motion is concluded from this method of adaptation. This shows the uniqueness of the circle among all possible geometrical forms. Adaptation of arbitrary motions using a circle neither helps nor does it harm. Adaptation using a circle is equal to no adaptation. If there is no adaptation of any motion it is the same as an adaptation with a circle. Consequently, the circle is worthless in the context of orbital mechanics. (Note: this is a mathematical issue only, not a physical issue). Therefore the circle is inviolable, it is sacrosanct. An ancient question is answered.

Thales of Miletus was not a natural philosopher. As one of the most highly regarded wise men of old, he endeavoured to find simple explanations for the world, however, he knew that „*in the simplest things, as the circle, something astonishing and authentic will be contained*"[29].

Remark: In order to avoid any troubles in reading this paper it should be emphasized that the content is totally restricted to a mathematical representation of the orbits. Of course, by physical-technical means a true circular motion can be controlled, as well as any motion can escape from circular motion. However this physical-technical fact is not subject of the present paper.

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[29] JAAP MANSFELD (1987), p.42

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