Rectilinear Motion

A Basis for the Adaptation of Arbitrary Motions

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Summary: Rectilinear motion will be used for the adaptation of any motion based on the method of the variation of the parameters. The formulae will be derived with respect to *Hansen's* "Ideal" coordinates. Examples show the use of the method. Based on rectilinear motion, a generalization of *Kepler*'s distance law is established which is valid for any motion.

Keywords: Rectilinear motion, adaptation of any motion, variation of parameters, *Lagrange* constraint, *Keplerian* distance law, general distance law, *Hansen* ("Ideal") frame, *Leibniz* frame

Table of Symbols:

A_R	equal area for solar radiation [m^2]
AU	astronomical unit (1.49597870×10^8 [km])
b_R, b_T, b_N	accelerations in radial, transversal, normal directions [km/s^2]
b_{R2}	radial "perturbation" of a 2-body motion [km/s^2]
b_{RC}	constant radial acceleration [km/s^2]
b_{RK}	*Keplerian* acceleration [km/s^2]
b_{SR}	radial acceleration due to thruster impuls [km/s^2]
$\mathbf{c}_1$, $\mathbf{c}_2$, C_1, C_2	integration constants of rectilinear motion (vector, scalar resp.)
c_R	reflectivity coefficient
d	distance along curve for equal area law [km]
e	numerical eccentricity of conic section
G	equal area parameter [km^2/s]
g	auxiliary factor g=s/V [sec]
g_1, g_2	parameter vector components, $g_1 = V\cos\zeta_P$, $g_2 = V\sin\zeta_P$ [km/s]

$\mathbf{p}_i$	Basis vectors of inertial system (*Newton* frame)
P_S	Solar pressure [kg/(ms^2)]
p	semilatus rectum of conic section [km]
$\mathbf{r}_0, \mathbf{q}_0, \mathbf{c}_0$	Basis vectors of co-moving system (orbit system, *Leibniz* frame)
r, r_P	radius, radius of pericentre distance [km]
r_K	radius of a circular orbit [km]
$\mathbf{q}_j^{(I)}$	Basis vectors of Ideal system (*Hansen* frame)
s	Distance along curve [km]
t, t_P	time, time of pericentre passage [s]
V	velocity [km/s]
ζ	orbit angle (first *Hansen* angle) [rad]
ζ_P	orbit angle (*Hansen* angle) [rad] of pericentre
η	spatial rotation angle (second *Hansen* angle) [rad]
μ	centric gravitational constant [km^3/s^2]
$\mu_\odot$	heliocentric gravitational constant ($\mu_\odot = 1.32712438 \times 10^{20}\ m^3/s^2$)

1 The Idea of Rectilinear Motion

Xenophanes of Kolophon (Ξενοφάνης, ca. 570 - 475 B.C.) seems to be the first, and probably the only, of the Pre-Socratian philosophers to assume a rectilinear motion of the celestial bodies

> Ξενοφάνης ..... τὸν ἥλιον εἰς ἄπειρον μὲν προιέναι, δοκεῖν δὲ κυκλεῖσθαι διὰ τὴν ἀπόστασιν
> *("Xenophanes ...says, the Sun would move to the infinite, however, due to the great distance its motion would appear to be circular")*[1].

He assumed that the Sun, the Moon and the stars were glowing clouds. They will be newly produced every day in the morning and they will cease glowing in the evening. All other Pre-Socratian philosophers seemed to believe in circular motion of the celestial bodies, an idea going back to *Anaximander of Miletus*.

Aristotle (Ἀριστοτέλης, 384 - 322 B.C.) assumes three basic motions: a rectilinear motion towards the centre of the Earth, a rectilinear motion away from the centre of the Earth, and a circular motion. All other motions will be composed of these kinds of motion:

> πᾶσα δὲ κίνησις ὅση κατὰ τόπον, ἣν καλοῦμεν φοράν, ἢ εὐθεῖα ἢ κύκλῳ ἢ ἐκ τούτων μικτή. ἁπλαῖ γὰρ αὗται δύο μόναι

[1] [1] fr. R 13, [2] DK 21A41a

(„Because each motion in space, we call it to carry, will be rectilinear or circular or mixed of both of them. Because these two are the only simple motions“)[2].

Aristotle again justifies this statement:

αἴτιον δ’ὅτι καὶ τὰ μεγέθη ταῦτα ἁπλᾶ μόνον, ἥ τ’εὐθεῖα καὶ ἡ περιφερής
(„The reason is why only these two parameters are simple, the rectilinear and what is moved in a circle“)[3].

Aristotle believes the circular motion to be the primary of all motions, in contrast to rectilinear motion:

’Αλλὰ μὴν καὶ πρώτην γε ἀναγκαῖον εἶναι τὴν τοιαύτην φοράν. τὸ γὰρ τέλειον πρότερον τῇ φύσει τοῦ ἀτελοῦς, ὁ δὲ κύκλος τῶν τελείων, εὐθεῖα δὲ γραμμὴ οὐδεμία
("In addition, such a motion will necessarily be the first. The reason for this is that it is natural that the perfect will be earlier than the imperfect. The circle is part of perfect things, but the rectilinear line will never be perfect. “)[4].

By way of these statements, *Aristotle* became responsible for thousands of years preference for circular motion.

In **a** mathematical sense, **Proclus** (412 - 485 A.D.) linked circular and rectilinear motion by the so-called Proclus device (cf. Figure 1), which was later (in the 13th century) applied by $Nas\bar{i}r\ ad-D\bar{i}n\ at-\d{T}\bar{u}s\bar{i}$ as “Tusi’s device”, in order to improve the theory of planetary latitudes originated by *Ptolemy*.

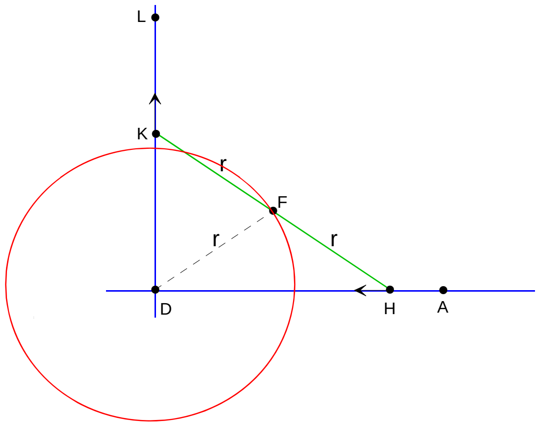

Figure 1: The Theorem of *Proclus* („Proclus’ device“): When H moves on the line DA towards the centre D and K is moving in the same time on the line DL towards L, then F, which is in the middle of the line HK, is moving on a circle centred on D

In a philosophical sense, **Nikolaus of Kues** (known as *Cusanus*, 1401-1464, A.D.) linked circular and rectilinear motion by his method of “Coincidentia Oppositorum” (= the coincidence of opposite extremes). As an example, he used a circle with infinite radius, which will be a rectilinear line.

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[2] [3] de caelo A II, 268b17

[3] [3] de caelo A II, 268b19

[4] [3] de caelo A II, 269 a 19-21

In a mathematical sense, this idea of *Cusanus* was reflected in the theory of rotations developed by **Benjamin Olinde Rodrigues** (1795–1850). In his theory, the *Euler*ian division of any motion into a rotational and a translatorial part is replaced by two consecutive rotations: instead of the translation, *Rodrigues* assumes a rotation with respect to a central axis in infinite distance, and perpendicular to the direction of motion[5].

A physical interpretation of the rectilinear motion was given by **Galileo Galilei** (1564-1642) by his detection of the law of inertia:

> *„A body is moving uniformly and rectilinearly if there is no force acting on it“.*

In his publication *Dialog*, there is the following citation of the motion of a body catapulted around and then released[6]:

> *1. „The motion impressed on the body, free flying after separation, can only be a rectilinear motion“*
>
> *2. „A thrown body moves along the tangent of the circle at the point of separation“.*
>
> *3. „Immediately after separation, the thrown body will have another direction of motion as the thrown body had before separation“.*

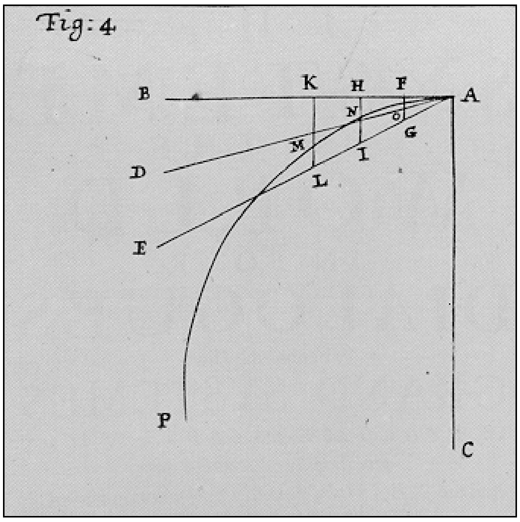

Figure 2: *Galileo*'s drawing describing the motion of a throw (copy from *Galileo's dialog*[7])

In contrast to planetary motion, **Johannes Kepler** (1751 – 1630) assumed a nearly rectilinear motion for the comets[8]. However, implicit in his second law, the cross-product of the position and velocity vector permits the adaptation of motion at the position described by the position vector by a rectilinear motion along the tangent in direction of the velocity vector.

It can be shown that no circle is able to be adapted to any motion[9]. In the following sections, however, it will be shown that a straight line can be adapted to any motion.

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5 [14], p. 375

6 see [4], p.120

7 copy from [4], p.120

8 see [10], p.288

9 see [5]

2 A Mathematical Description of Rectilinear Motion

2.1 The Basic Equations

Based on the law of inertia, uniform rectilinear motion will be possible if there are no perturbations acting on the moving body. Therefore from

$$\ddot{\mathbf{r}} = 0 \tag{1}$$

two integrations with respect to time t in an inertial space frame lead to the position vector

$$\mathbf{r} = \mathbf{c}_1\, t + \mathbf{c}_2 \quad . \tag{2}$$

In this solution, the first constant is identical to the velocity vector

$$\mathbf{c}_1 = \dot{\mathbf{r}} \quad , \tag{3}$$

the second constant vector $\mathbf{c}_2$ marks an initial position along the orbit.

Relating to a co-moving rectangular coordinate system with direction vectors $\mathbf{r}_0$ in the radial, $\mathbf{q}_0$ in the transversal ("transradial"), and $\mathbf{c}_0$ in the normal direction, the acceleration vector describing the motion can be written in the form

$$\ddot{\mathbf{r}} = b_R\,\mathbf{r}_0 + b_T\,\mathbf{q}_0 + b_N\,\mathbf{c}_0 \quad . \tag{4}$$

According to the accompanying trihedron[10], $\mathbf{r}_0$, $\mathbf{q}_0$, $\mathbf{c}_0$, the acceleration is divided in the radial b_R, the transversal b_T and the normal b_N acceleration. If r is the radius of the moving body and

$$\mathbf{r} = r\,\mathbf{r}_0 \tag{5}$$

is the position vector, then the velocity vector is given by

$$\dot{\mathbf{r}} = \dot{r}\,\mathbf{r}_0 + r\,\dot{\mathbf{r}}_0 = \dot{r}\,\mathbf{r}_0 + r\,\dot{\zeta}\,\mathbf{q}_0 \quad . \tag{6}$$

If, and only if, $\dot{\zeta} := |\dot{\mathbf{r}}_0|$ is assumed to be the value of the variation of the radial direction vector $\mathbf{r}_0$, then ζ can be shown to be an orbit angle related to the motion oriented *Hansen* ("Ideal") coordinate system[11]. (A *Hansen* system is a coordinate system in which the velocity vector is independent of the proper motion of this system). This leads to the orbit normal vector[12]

$$\mathbf{c} = \mathbf{r} \times \dot{\mathbf{r}} = r^2\,\dot{\zeta}\,\mathbf{c}_0 = G\,\mathbf{c}_0 \quad , \quad \left(\mathbf{c}_0{}^2 = 1, G = |\mathbf{c}| \geq 0\right) \quad , \tag{7}$$

where the general relation $r^2\,\dot{\zeta} = G$ is completely independent of any orbit form. Because $G \geq 0$, this relation includes the following issue: if the parameter ζ is interpreted as usual as an orbit angle, its direction of variation must always be positive[13]. The equation (7) corresponds to the second *Kepler*ian law if, and only if, the equal area parameter G[km^2/s] is a

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[10] As firstly introduced into celestial mechanics by *G.W. Leibniz* [7], therefore, this system is sometimes called "*Leibniz* frame"

[11] see [6], theorem 13, advice also in [19], p.69

[12] the value of the orbit normal vector will be designed with the symbol G according to the corresponding canonical element in *Ch. Delaunay*'s theory and is in this form already widely used in Astrodynamics

[13] cf. e.g. [17], p. 46

constant. The acceleration related to the co-moving coordinate system ("orbit system", "*Leibniz* frame") is

$$\ddot{\mathbf{r}} \;=\; \ddot{r}\,\mathbf{r}_0 + 2\,\dot{r}\,\dot{\zeta}\,\mathbf{q}_0 + r\,\ddot{\zeta}\,\mathbf{q}_0 + r\,\dot{\zeta}\,\dot{\mathbf{q}}_0 \quad . \tag{8}$$

Introducing parameters $\dot{\xi}, \dot{\eta}$ and because $\mathbf{q}_0{}^2 = 1$, the variation of the transversal direction vector $\mathbf{q}_0$ can only take place in a plane perpendicular to this vector

$$\dot{\mathbf{q}}_0 \;=:\; \dot{\xi}\,\mathbf{r}_0 + \dot{\eta}\,\mathbf{c}_0 \quad . \tag{9}$$

The motion acceleration vector is then

$$\ddot{\mathbf{r}} \;=\; \left(\ddot{r} + r\,\dot{\zeta}\,\dot{\xi}\right)\mathbf{r}_0 + \left(2\,\dot{r}\,\dot{\zeta} + r\,\ddot{\zeta}\right)\mathbf{q}_0 + r\,\dot{\zeta}\,\dot{\eta}\,\mathbf{c}_0 \quad . \tag{10}$$

Therefore, by comparing equations (4) and (10), and applying (7), the transversal acceleration leads to the variational equation for the parameter G based on the transversal acceleration only

$$\dot{G} \;=\; r\,b_T \quad . \tag{11}$$

The variational equation for the parameter η is based on the normal acceleration only

$$\dot{\eta} \;=\; \frac{r}{G}\,b_N \quad . \tag{12}$$

It can be shown[14] that $\dot{\xi} = -\dot{\zeta}$ is a consequence of the relation to a *Hansen* system, therefore, the general equation of motion can be written in the form

$$\ddot{\mathbf{r}} \;=\; \left(\ddot{r} - \frac{G^2}{r^3}\right)\mathbf{r}_0 + \frac{\dot{G}}{r}\,\mathbf{q}_0 + \frac{G}{r}\,\dot{\eta}\,\mathbf{c}_0 \quad . \tag{13}$$

Comparison with equation (4) finally leads to the "general *Leibniz* equation"

$$\ddot{r} \;=\; \frac{G^2}{r^3} + b_R \quad . \tag{14}$$

In case of the two body problem, *G. W. Leibniz*[15] derived this equation based on the Keplerian acceleration $b_R \sim -1/r^2$ and taking into account the first two *Kepler*ian laws[16].

If no acceleration acts on the motion, then $b_R = b_T = b_N = 0$ and G = const. and η = const. (= 0). In this case, the *Leibniz* equation leads to the first integral

$$\dot{r} \;=\; \pm\sqrt{C_1^2 - \frac{G^2}{r^2}} \quad . \tag{15}$$

The new constant C_1[km/s] has to fulfil the condition

$$r \;\geq\; \frac{G}{C_1} \;\geq\; 0 \quad . \tag{16}$$

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[14] see [6] section 4, formula (57) and theorem 15

[15] see [7]

[16] cf. [8], p. 67

Following equation (7), G is non-negative and C_1 must be positive. Therefore the auxiliary parameter

$$g \quad := \quad \sqrt{r^2 C_1^2 - G^2} \tag{17}$$

can be used. A geometrical interpretation of this parameter is possible according to Figure 3: g / C_1 is the (arc-) length along the (rectilinear) orbit as counted from the orbit point next to the initial point P (the „pericentre“ of the rectilinear orbit) and the position r of the orbit. Because

$$g\dot{g} \quad = \quad C_1^2 \, r \, \dot{r} \quad , \tag{18}$$

and, using the differential equation (15), it follows that

$$r\dot{r} \quad = \quad \sqrt{r^2 C_1^2 - G^2} \quad = \quad g \quad , \tag{19}$$

therefore,

$$\dot{g} = C_1^2 \quad . \tag{20}$$

This leads to the integral

$$g \quad = \quad C_1^2 \, t + C_2 \quad . \tag{21}$$

C_2[km^2/s] is the new integrational constant. The radius will be calculated from equation (17):

$$r \quad = \quad \sqrt{C_1^2 t^2 + 2 C_2 \, t + \frac{C_2^2 + G^2}{C_1^2}} \quad . \tag{22}$$

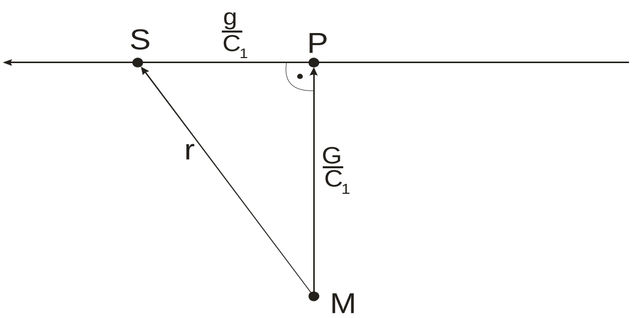

Figure 3: Diagram of the rectilinear motion

In orbital mechanics, the orbit angle ζ usually replaces the time t by means of equation (7)

$$d\zeta \quad = \quad \frac{G}{r^2} dt \quad = \quad \frac{G \, dt}{C_1^2 t^2 + 2 C_2 \, t + \frac{C_2^2 + G^2}{C_1^2}} \tag{23}$$

with the integral

$$\zeta - \zeta_P \quad = \quad \arctan\left[\frac{1}{G} \left(C_1^2 \, t + C_2 \right) \right]$$

or

$$\tan(\zeta - \zeta_P) \;=\; \frac{1}{G}\left(C_1^2\, t + C_2\right) \;=\; \frac{s}{G} \;=\; \frac{\frac{s}{C_1}}{\frac{G}{C_1}} \quad . \tag{24}$$

Applying equation (17), we have

$$\tan(\zeta - \zeta_P) \;=\; \frac{1}{G}\sqrt{C_1^2\, r^2 - G^2} \quad . \tag{25}$$

The new integrational constant ζ_P can be related to C_2 by

$$\tan\left[\zeta(t=0) - \zeta_P\right] \;=\; \frac{C_2}{G} \quad , \quad \zeta_P \;:=\; \zeta(t = t_P) \quad . \tag{26}$$

This relation is geometrically interpreted in Figure 4.

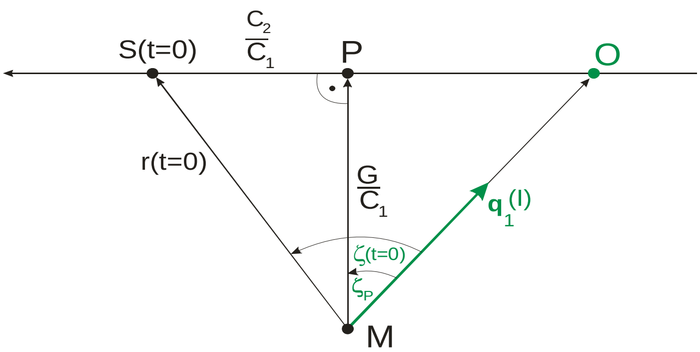

Figure 4: The relation of the parameter ζ_P and C_2 of the rectilinear motion

According to equation (24), the time t and orbit angle ζ are connected. The time can be derived from the orbit angle ζ via

$$t \;=\; \frac{1}{C_1^2}\left[G\tan(\zeta - \zeta_P) - C_2\right] \quad . \tag{27}$$

Because $C_1 > 0$, this relationship has no singularity. In order to relate the constant C_2 to the initial time $t_{P,}$ we define the pericentre distance at the moment of the pericentre pass as

$$r_P \;:=\; r(t_P) \;=\; \frac{G}{C_1} \quad . \tag{28}$$

Using equation (21) we have

$$C_2 \;=\; -C_1^2\, t_P \quad . \tag{29}$$

From equation (25), the expression for the radius with respect to the orbit angle ζ will be

$$r \;=\; \frac{G}{C_1\cos(\zeta - \zeta_P)} \quad . \tag{30}$$

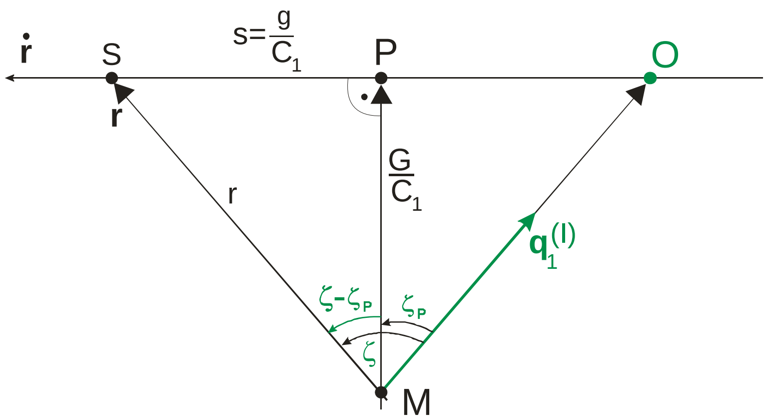

Figure 5: Rectilinear motion: The geometrical visualization of the mathematical description

This relation has no singularity since, with $G \geq 0$ and $C_1 > 0$, it is always the case that $\cos(\zeta - \zeta_P) > 0$ and the angle $\zeta - \zeta_P$ must lie in the interval

$$-90° \quad < \quad \zeta - \zeta_P \quad < \quad 90° \quad . \tag{31}$$

Figure 5 confirms that the singularity $|\zeta - \zeta_P| = 90°$ will never happen, whenever the origin M is not lying on the straight line of the rectinlinear motion.

The variation of the radius (i.e. the radial velocity) will follow from equation (15)

$$V_R \;=\; \dot{r} \;=\; C_1 \sin(\zeta - \zeta_P) \quad . \tag{32}$$

Based on equations (19), (21), (22) and (29), the following relation can be derived

$$r\,\dot{r} \;=\; G\tan(\zeta - \zeta_P) \;=\; C_1^2\, t + C_2 \;=\; C_1^2\,(t - t_P) \quad . \tag{33}$$

Finally, the transversal velocity will be found using equation (7), and taking into account equation (30):

$$V_T \;=\; r\,\dot{\zeta} \;=\; \frac{G}{r} \;=\; C_1 \cos\left(\zeta - \zeta_P\right) \quad . \tag{34}$$

With the radial and the transversal velocities derived in equations (32) and (34), the velocity vector (6) leads to

$$\dot{\mathbf{r}} \;=\; C_1\left[\mathbf{r}_0 \sin(\zeta - \zeta_P) + \mathbf{q}_0 \cos(\zeta - \zeta_P)\right] \quad .$$

If the absolute velocity is assumed to be $V = |\dot{\mathbf{r}}|$ then

$$\dot{\mathbf{r}}^2 \;=\; V^2 \;=\; C_1^2 \quad , \tag{35}$$

and because C_1>0, it is shown that the first constant of the rectilinear motion is identical to the velocity

$$C_1 \;=\; V \quad . \tag{36}$$

In conclusion, the state vector of the rectilinear motion is

$$\mathbf{r} \;=\; \frac{G}{V\cos\left(\zeta-\zeta_P\right)}\,\mathbf{r}_0 \quad , \quad \dot{\mathbf{r}} \;=\; V\left[\mathbf{r}_0\sin\left(\zeta-\zeta_P\right)+\mathbf{q}_0\cos(\zeta-\zeta_P)\right] \,, \tag{37}$$

the radius, radial, and transversal velocity are given by

$$\begin{aligned} r &= \frac{G}{V\cos\left(\zeta-\zeta_P\right)} \\ V_R &= \dot{r} \quad = V\sin\left(\zeta-\zeta_P\right) \\ V_T &= r\dot{\zeta} \quad = V\cos\left(\zeta-\zeta_P\right) \end{aligned} \tag{38}$$

and we find

$$r\,\dot{r} \;=\; V^2\,(t-t_P) \;=\; G\tan\left(\zeta-\zeta_P\right) \quad . \tag{39}$$

2.2 The Law of Areas in Rectilinear Motion

Kepler's second law is a general behaviour of any central motion as shown by *L. Euler*[17]. This law has no other content than to state that the behaviour of such a motion always takes place in a fixed plane. This will be represented by

$$\mathbf{r}\times\dot{\mathbf{r}} \;=\; \mathbf{c} \qquad , \qquad \mathbf{c} = \text{ const.}$$

We would like to investigate the area law in case of a (uniform) rectilinear motion. In this case of constant velocity, $V = const.$, the body will always travel in equal time intervals, Δt, the same distance

$$d \;=\; V\,\Delta t \quad . \tag{40}$$

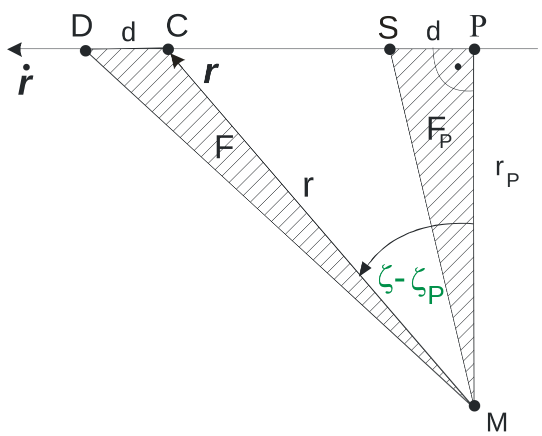

Figure 6: The area law for the case of uniform rectilinear motion

According to Figure 6, the area F_P formed by the triangle *MPS* (P should be the pericentre in relation to a fictive origin *M*) shall be set in relation to the area *F* of a triangle *MCD* at any

[17] published in *Mémoires de l'Académie des Sciences de Berlin*, **3**, 1749, pp. 93-143; cited in [9], p.346 pp.

point in time. Let $r_P = \overline{MP}$ be the pericentre distance, then the area of the triangle *MPS,* related to the distance passed in the given time unit, will be

$$F_P \;=\; \frac{1}{2} r_P\; d \quad .$$

r_P will be assumed the height of the triangle, the distance *d* its base. Then, the height of the triangle is always identical, so that with equation (40) and the identical base, the area of the triangle will not change, and consequently $F \;=\; F_P$.

The conclusion is the law going back to *Kepler*: the radius vector of a planar motion sweeps out equal areas in equal times.

2.3 The General Distance Law

The law of distance was detected by *Kepler* before he derived the area law. In the third part of his "battle against the planet Mars", *Kepler* determined the velocity of the planet in the apsides to be inverse to the distance from the Sun: $V \sim 1/r$. However, beyond the apsides, this relation does not hold. The transversal velocity is known by the relation $V_T = r\,\dot{\zeta}$ (see the third equation in (38)), whereas the area law is given in scalar form by $r^2\,\dot{\zeta} = G$ (see equation (7))[18]. Therefore,

$$V_T \;=\; r\dot{\zeta} = \frac{G}{r} \quad . \tag{41}$$

In this form, the distance law is valid for all motions, i.e. it is more general than the area law: the distance of a moving body is proportional to the inverse of the transversal velocity with respect to the origin of the system. Based on the polar equation in (38), we obtain for the absolute velocity

$$V = \frac{G}{r}\frac{1}{\cos\left(\zeta - \zeta_P\right)} = \frac{G\left(\zeta\right)}{r\left(\zeta\right)}\frac{1}{\cos\left(\zeta - \zeta_P\left(\zeta\right)\right)} \quad . \tag{42}$$

This equation is based on the formalism of rectilinear motion and is related to a *Hansen* system by means of the *Hansen* orbit angle ζ and the (*Hansen*-) pericentre angle ζ_P. It is of general validity and can be considered as a generalization of *Kepler*'s distance law. However, in contrast to the familiar *Kepler* motion, in this case, the pericentre angle ζ_P must <u>always</u> be related to the adapting rectilinear motion[19].

2.4 The Elementary Vectors of Rectilinear Motion

The *Hansen* coordinate system introduced by the orbit angle ζ in equation (6) will be defined by the basic vectors $\mathbf{q}_1^{(I)}, \mathbf{q}_2^{(I)}, \mathbf{q}_3^{(I)} = \mathbf{q}_1^{(I)} \times \mathbf{q}_2^{(I)}$. This system is related to the co-moving system via

$$\mathbf{r}_0 \;=\; \mathbf{q}_1^{(I)} \cos\zeta + \mathbf{q}_2^{(I)} \sin\zeta \quad , \tag{43}$$

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[18] see for reference: *Astronomia Nova*, Ch. 59, *Epitome* KGW (Kepler collected works) VII, pp.597-598, citation in [16], p. 90

[19] as demonstrated e.g. in Figure 10

and

$$\mathbf{q}_0 \quad = \quad -\mathbf{q}_1^{(I)} \sin\zeta + \mathbf{q}_2^{(I)} \cos\zeta \quad . \tag{44}$$

Additionally, the normal direction vector will be

$$\mathbf{c}_0 \quad = \quad \mathbf{r}_0 \times \mathbf{q}_0 \quad = \quad \mathbf{q}_3^{(I)} \quad = \quad \mathbf{q}_1^{(I)} \times \mathbf{q}_3^{(I)} \quad . \tag{45}$$

The direction of $\mathbf{q}_j^{(I)}$ is defined by the initial direction of the orbital angle $\zeta = 0°$, as shown in Figure 7.

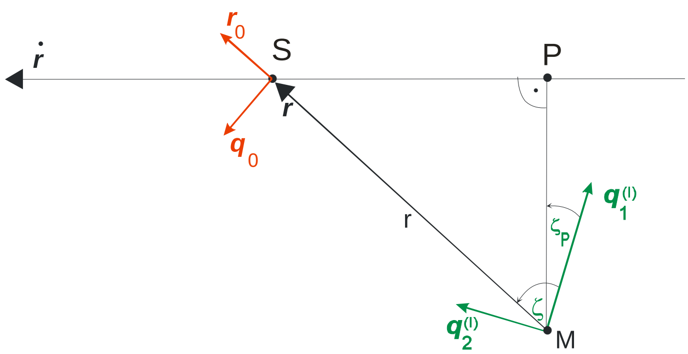

Figure 7: The basic vectors in the Hansen system of rectilinear motion

In order to compare the relative motion (5), (6) $\mathbf{r} = r\,\mathbf{r}_0$, $\dot{\mathbf{r}} = \dot{r}\,\mathbf{r}_0 + r\,\dot{\zeta}\,\mathbf{q}_0$ with the inertial solution (2), (3) $\mathbf{r} = \mathbf{c}_1 t + \mathbf{c}_2$, $\dot{\mathbf{r}} = \mathbf{c}_1$ of the rectilinear motion, we have

$$\mathbf{c}_1 \quad = \quad V\left(-\mathbf{q}_1^{(I)} \sin\zeta_P + \mathbf{q}_2^{(I)} \cos\zeta_P\right) \tag{46}$$

and

$$\mathbf{c}_2 \quad = \quad \mathbf{q}_1^{(I)}\left(r\cos\zeta + t\,V\sin\zeta_P\right) + \mathbf{q}_2^{(I)}\left(r\sin\zeta - t\,V\cos\zeta_P\right) \quad .$$

Since $\mathbf{c_2}$ = *const.* we can set $t = t_P$ = *const.*. Then, based on equations (38) and using $\zeta_P = \zeta\left(t_P\right)$ for the pericentre distance, we have

$$r_P \quad = \quad r\left(t = t_P\right) \quad = \quad \frac{G}{V} \qquad \text{and} \qquad C_2 \quad = \quad -t_P V^2 \quad . \tag{47}$$

For the initial position the constant vector remains (cf. Figure 8)

$$\mathbf{c}_2 \quad = \quad \mathbf{q}_1^{(I)}\left(\frac{G}{V}\cos\zeta_P - \frac{C_2}{V}\sin\zeta_P\right) + \mathbf{q}_2^{(I)}\left(\frac{G}{V}\sin\zeta_P + \frac{C_2}{V}\cos\zeta_P\right) \quad . \tag{48}$$

Vice versa, the basic vectors $\mathbf{q}_j^{(I)}$ of the motion related *Hansen* system will be related to the integrational constants of the rectilinear motion via the equations

$$
\begin{aligned}
\mathbf{q}_1^{(I)} &= -\mathbf{c}_1\left(\frac{1}{V}\sin\zeta_P + \frac{C_2}{V\,G}\cos\zeta_P\right) + \mathbf{c}_2\,\frac{V}{G}\cos\zeta_P \\
\mathbf{q}_2^{(I)} &= \mathbf{c}_1\left(\frac{1}{V}\cos\zeta_P - \frac{C_2}{V\,G}\sin\zeta_P\right) + \mathbf{c}_2\,\frac{V}{G}\sin\zeta_P \\
\mathbf{q}_3^{(I)} &= \frac{1}{G}\,\mathbf{c}_2\times\mathbf{c}_1 \quad .
\end{aligned}
\tag{49}
$$

In the case of an „unperturbed“ uniform rectilinear motion with parameters G = *const.*, $\mathbf{c_1}$ = *const.*, $\mathbf{c_2}$ = *const.* resp. V = *const.*, ζ_P = *const.* , $\mathbf{q}_j^{(I)}$ will be constant. Therefore, the system proper motion vector $\mathbf{D}_{q^{(I)}}$ (the “system *Darboux* vector”) will fulfil the condition

$$
\dot{\mathbf{q}}_j^{(I)} = \mathbf{D}_{q^{(I)}}\times\mathbf{q}_j^{(I)} = 0 \qquad (\dot{\mathbf{r}} = const.) \quad \Leftrightarrow \quad \mathbf{D}_{q^{(I)}} = 0 \quad . \tag{50}
$$

▶ *In (uniform) rectilinear motion, the* Darboux *vector (“absolute system proper motion vector”) will be the zero vector. There is no proper motion of the basic* Hansen *system.*

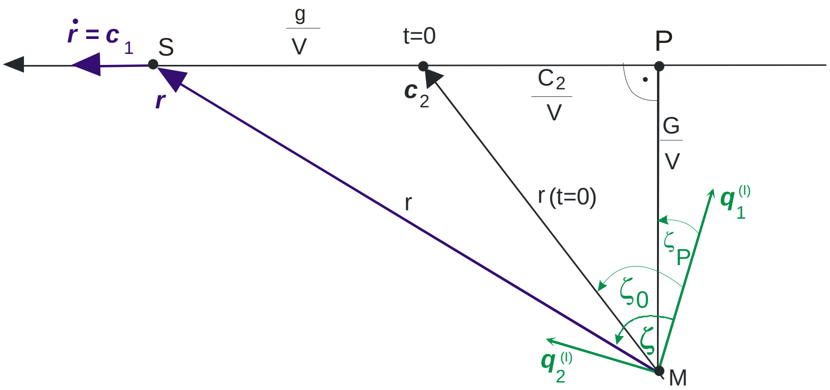

Figure 8: Geometrical interpretation of the integrational constants of rectilinear motion

Consequently, the basic vectors of the co-moving system will fulfil the trihedron equations

$$
\begin{aligned}
\dot{\mathbf{r}}_0 &= \frac{G}{r^2}\,\mathbf{q}_0 \\
\dot{\mathbf{q}}_0 &= -\frac{G}{r^2}\,\mathbf{r}_0 \qquad (\dot{\mathbf{r}} = const.) \\
\dot{\mathbf{c}}_0 &= 0 \quad .
\end{aligned}
\tag{51}
$$

Using the orbit angle ζ instead of the time t, according to equation (7), these equations can be simplified still further:

$$\begin{aligned} \frac{d\mathbf{r}_0}{d\zeta} &= \mathbf{q}_0 \\ \frac{d\mathbf{q}_0}{d\zeta} &= -\mathbf{r}_0 \qquad (\dot{\mathbf{r}} = const.) \\ \frac{d\mathbf{c}_0}{d\zeta} &= 0 \quad . \end{aligned} \tag{52}$$

3 Adaptation of Rectilinear Motion to any other Motion

The adaptation of any motion by use of a curve of the first order will be a very interesting example for the general use of the method of the variation of the parameters. Any motion will only be modified by accelerations (the physical reason for any acceleration is of no interest in the frame of a mathematical treatment). According to the method of the variation of the parameters, the velocity V and the initial angle ζ_P (or corresponding parameters of the rectilinear motion as we will see later) have to be assumed as variable parameters. These parameters have to be varied in such a way that the real motion, as performed according to the acting accelerations, will be described based on the equations of rectilinear motion. Consequently, the corresponding variational equations of the characterizing parameters of the rectilinear motion have to be found.

3.1 Time as an Independent Variable

For any integration of the equations of any motion the time dependent variational equations(14), (11), (12) will be used instead of the vectorial equation (13). The time relation of an orbital trajectory will be established by the equal area law (7). We have the following general set of equations:

$$\ddot{r} = \frac{G^2}{r^3} + b_R \ , \ \dot{G} = r\, b_T \ , \dot{\eta} = \frac{r}{G} b_N \ , \ r^2 \dot{\zeta} = G \quad . \tag{53}$$

An analytical investigation of the behaviour of any motion based on the formulation of a rectilinear motion will be derived in the following way.

The time dependency of the radius of the moving particle is given by equation (22). Differentiation leads to

$$r\dot{r} = \dot{V}\left(V t^2 - \frac{C_2^2 + G^2}{V^3}\right) + \dot{C}_2\left(t + \frac{C_2}{V^2}\right) + \dot{G}\frac{G}{V^2} + V^2 t + C_2 \quad . \tag{54}$$

Adaptation of any motion by a curve of the first order, requires at any instant formal identity of the equation of a straight line according to equation (22) and of the radial velocity according to the second of the equations (38):

$$r\dot{r} = V^2 t + C_2 \quad . \tag{55}$$

Therefore, the adaptation necessarily requires validity of the conditional equation

$$\dot{V}\left(V t^2 - \frac{C_2^2 + G^2}{V^3}\right) + \dot{C}_2\left(t + \frac{C_2}{V^2}\right) + \dot{G}\frac{G}{V^2} = 0 \quad . \tag{56}$$

This is the only place where the procedure of the adaptation has to be applied. This equation will therefore be called "equation of osculation" or "first equation of adaptation". It can be seen by this procedure that the "real" curve has nothing to do with the curve of adaptation. This curve will be chosen completely arbitrarily. The central request in the method of adaptation will be the correct use of the physical accelerations acting on the moving body.

Because of the variational equation for the area parameter from equation (11) $\dot{G} = r\,b_T$, the first equation for the adaptation will be reduced to

$$\dot{C}_2\,(V^2 t + C_2) \;=\; -\dot{V}\left(V^3 t^2 - \frac{C_2^2 + G^2}{V}\right) - r\,G\,b_T \quad . \tag{57}$$

This equation contains two parameters to be varied. Thus a second equation of adaptation has to be found. To this purpose, the radial acceleration will be calculated applying equation (55)

$$\ddot{r} \;=\; \frac{1}{r}(2V\dot{V}\,t + \dot{C}_2) + \frac{V^2}{r} - \frac{\dot{r}^2}{r} \quad . \tag{58}$$

With V=$C_{1,}$ equation (15) leads to

$$\frac{\dot{r}^2}{r} \;=\; \frac{V^2}{r} - \frac{G^2}{r^3} \tag{59}$$

and so

$$\ddot{r} \;=\; \frac{1}{r}(2V\dot{V}\,t + C_2) + \frac{G^2}{r^3} \quad . \tag{60}$$

Application of the *Leibniz* Equation (14) leads to the second conditional equation to obtain variational equations for $\dot{V}$ und $\dot{C}_2$

$$2V\dot{V}\,t + \dot{C}_2 \;=\; r\,b_R \quad . \tag{61}$$

Therefore, the variational equations for the parameters V and C_2 are

$$\begin{aligned} \dot{V} &= \frac{(V^2 t + C_2)\,b_R + G\,b_T}{\sqrt{(V^2 t + C_2)^2 + G^2}} \\ V\,\dot{C}_2 &= \frac{(C_2^2 + G^2 - V^4 t^2)\,b_R - 2V^2 G\,b_T}{\sqrt{(V^2 t + C_2)^2 + G^2}} \quad . \end{aligned} \tag{62}$$

The initial time t_P can be calculated with relation (29)

$$\frac{dt_P}{dt} \;=\; \frac{1}{V^3}\left(2C_2\dot{V} - V\dot{C}_2\right)$$

or

$$V^3\frac{dt_P}{dt} \;=\; \frac{\left[(V^2 t + C_2)^2 - G^2\right] b_R + 2G\,(V^2 t + C_2)\,b_T}{\sqrt{(V^2 t + C_2)^2 + G^2}} \quad . \tag{63}$$

The solutions of the above process are the parameters as explicit functions of time: $V(t)$,$C_2(t)$, $t_P(t)$. Finally, the orbit angle ζ will be obtained as a function of time by means of the variational equation (7)

$$\zeta \quad = \quad \zeta_P + \int_{t_P}^{t} \frac{V^2 G}{\left(V^2 t + C_2\right)^2 + G^2} \, dt \quad . \tag{64}$$

3.2 The (*Hansen*-) Orbit Angle as an Independent Variable

The use of the orbit angle ζ (“first *Hansen* angle”) instead of the time t allows a more elegant solution to the problem of celestial motions. The time t is only and only related to the *Hansen* orbit angle[20] by the equal area law $r^2 \dot{\zeta} = G$. For the case of rectilinear motion, equation (30) leads to

$$\dot{r} \quad = \quad \frac{r}{G}\left[\dot{G} - \frac{G}{V}\dot{V} - rV\sin\left(\zeta - \zeta_P\right)\dot{\zeta}_P\right] + \frac{G\sin\left(\zeta - \zeta_P\right)}{V\cos^2\left(\zeta - \zeta_P\right)}\dot{\zeta} \quad .$$

Adaptation of any motion based on rectilinear motion requires the unchanged radial velocity from (38)

$$\dot{r} \quad = \quad V\sin\left(\zeta - \zeta_P\right) \quad . \tag{65}$$

This will be accomplished by the Lagrange constraint

$$\frac{dr}{dt} = \frac{\partial r}{\partial t} \quad , \tag{66}$$

which is a consequence of the behaviour of a *Hansen* system (see [18], theorem 20 on page 123). The first equation of adaptation is thus necessarily

$$\dot{G} - \frac{G}{V}\dot{V} - rV\sin\left(\zeta - \zeta_P\right)\dot{\zeta}_P \quad = \quad 0 \quad . \tag{67}$$

The derivative of the radial velocity (65) leads to the radial acceleration

$$\ddot{r} \quad = \quad \frac{G^2}{r^3} + \frac{V\sin\left(\zeta - \zeta_P\right)}{G}\dot{G} - \frac{rV^2}{G}\dot{\zeta}_P \tag{68}$$

and comparison with the *Leibniz* equation (14) leads finally to the second equation of adaptation

$$V\dot{G}\sin\left(\zeta - \zeta_P\right) - r\,V^2\dot{\zeta}_P - G\,b_R \quad = \quad 0 \quad . \tag{69}$$

With $\dot{G} = r\,b_T$ the variational equation for the pericentre angle parameter ζ_P will be obtained.

$$V\,\dot{\zeta}_P \quad = \quad -b_R\cos\left(\zeta - \zeta_P\right) + b_T\sin\left(\zeta - \zeta_P\right) \quad . \tag{70}$$

By means of the first equation of adaptation, the variational equation for the „parameter“ V, the velocity of the moving body, applies

$$\dot{V} \quad = \quad b_R\sin\left(\zeta - \zeta_P\right) + b_T\cos\left(\zeta - \zeta_P\right) \quad . \tag{71}$$

---

[20] See [18] theorem 9, p. 38; in [19] the angle ζ is related to a *Hansen* ideal coordinate system without proof (there designed by the letter θ, cf. formula (12) on page 69)

A geometrical interpretation of these varying parameters can be seen in Figure 9: $\dot{V}$ is acting in the direction of the instantaneous motion, $V\,\dot{\zeta}_P$ across the direction of motion, causing a rotation of the motion with respect to the origin *O* of the system.

Based on the orbit angle ζ as independent variable, the two variational equations apply

$$\begin{aligned} V\,\frac{d\zeta_P}{d\zeta} &= \frac{r^2}{G}\left[-b_R\cos\left(\zeta-\zeta_P\right)+b_T\sin\left(\zeta-\zeta_P\right)\right] \\ \frac{dV}{d\zeta} &= \frac{r^2}{G}\left[\;\;b_R\sin\left(\zeta-\zeta_P\right)+b_T\cos\left(\zeta-\zeta_P\right)\right] \end{aligned} \qquad (72)$$

After calculation of the instantaneous parameters $V=V(\zeta)$ und $\zeta_P=\zeta_P\left(\zeta\right)$, the relation with time will be obtained by applying the variational equation

$$dt \;=\; \frac{r^2}{G}d\zeta$$

by the integral (which is the analogue to the *Kepler* Equation in classical celestial mechanics)

$$t \;=\; t_0+\int_{\zeta_0}^{\zeta}\frac{G}{V^2\cos^2\left(\zeta-\zeta_P\right)}d\zeta \quad . \qquad (73)$$

ζ_0 and t_0 are initial constant values. With

$$t_P \;=\; t\left(\zeta=\zeta_P\right) \qquad (74)$$

the parameter C_2 will be obtained using the equation (29)

$$C_2 \;=\; -V^2 t_P \quad . \qquad (75)$$

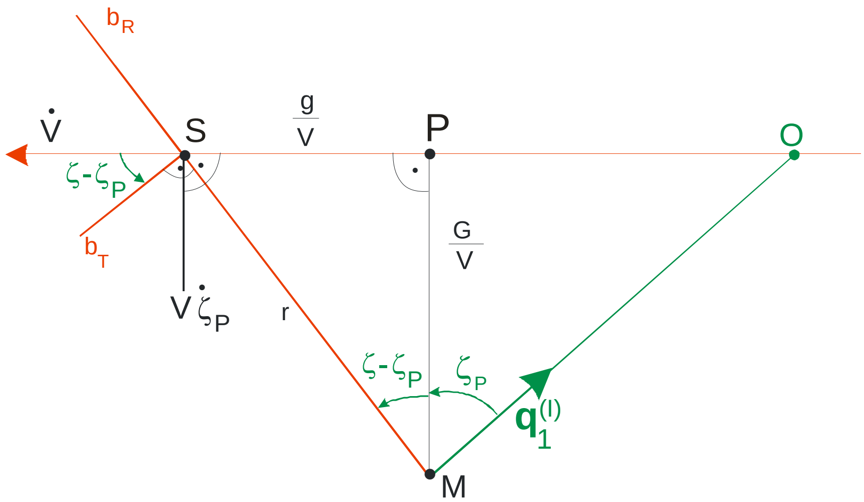

Figure 9: A geometrical interpretation of the accelerations acting on the moving body in case of adaptation using a rectilinear curve

A complete solution of any problem of motion requires the state vector $\mathbf{r},\dot{\mathbf{r}}$. Dependent on the orbit angle ζ and, with relation to the *Hansen* (- Ideal) $\mathbf{q}_j^{(I)}$ – orbit related coordinate system, we have

$$\begin{aligned}\mathbf{r}_0 &= \mathbf{q}_1^{(I)}\cos\zeta + \mathbf{q}_2^{(I)}\sin\zeta \\ \mathbf{q}_0 &= -\mathbf{q}_1^{(I)}\sin\zeta + \mathbf{q}_2^{(I)}\cos\zeta \quad ,\end{aligned} \tag{76}$$

and

$$\dot{\mathbf{r}} \;=\; \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\zeta}\frac{\mathrm{G}}{\mathrm{r}^2} \quad . \tag{77}$$

Finally,

$$\begin{aligned}\mathbf{r} \quad &= \; \mathrm{r}\,\mathbf{r}_0 \\ \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\zeta} \;&=\; \frac{\mathrm{dr}}{\mathrm{d}\zeta}\mathbf{r}_0 \;+\; \mathrm{r}\,\mathrm{G}\,\mathbf{q}_0\end{aligned} \tag{78}$$

will completely describe the adapted motion.

3.3 The Leibniz equation in the *Hansen* System

For some applications, especially as in the case of a numerical solution of a given problem of motion, the *Leibniz* equation will needed in dependency of the *Hansen* orbit angle ζ. Based on the equations (53) we have the relations

$$\frac{dr}{d\zeta} = \frac{dr}{dt}\frac{dt}{d\zeta} = \frac{\dot r}{\dot\zeta} = \dot r\frac{r^2}{G} \quad , \quad \frac{dG}{d\zeta} = \frac{r^3}{G}b_T \quad ,$$

$$\frac{d^2r}{d\zeta^2} = \frac{d\dot r}{d\zeta}\frac{r^2}{G} + \dot r\frac{2r}{G}\frac{dr}{d\zeta} - \frac{\dot r\, r^2}{G^2}\frac{dG}{d\zeta} = \frac{d\dot r}{dt}\frac{d\zeta}{dt}\frac{r^2}{G} + \frac{2r}{G}\frac{G}{r^2}\frac{dr}{d\zeta}\frac{dr}{d\zeta} - \frac{dr}{d\zeta}\frac{1}{G}\frac{dG}{d\zeta}$$

$$\frac{d^2r}{d\zeta^2} = \ddot r\frac{r^4}{G^2} + \frac{2}{r}\left(\frac{dr}{d\zeta}\right)^2 - \frac{r^3}{G^2}b_T\frac{dr}{d\zeta} \quad .$$

Finally with $\ddot r$ from the set (53) we have the *Leibniz* equation related to the *Hansen* system:

$$\frac{d^2r}{d\zeta^2} - \frac{2}{r}\left(\frac{dr}{d\zeta}\right)^2 + \frac{r^3}{G^2}b_T\frac{dr}{d\zeta} - r - \frac{r^4}{G^2}b_R = 0 \quad . \tag{79}$$

3.4 Adaptation of any Motion using a Curve of the First Order in the *Hansen* System

In the presentation of the state vector

$$\begin{aligned}\mathbf{r} \;&=\; r\,\mathbf{r}_0 \\ \dot{\mathbf{r}} \;&=\; \dot r\,\mathbf{r}_0 + r\,\dot\zeta\,\mathbf{q}_0\end{aligned} \tag{80}$$

r and ζ are the polar coordinates related to the instantaneous orbital plane of the moving body. $r\cos\zeta$ and $r\sin\zeta$ are the corresponding Cartesian coordinates in the motion related $\mathbf{q}_j^{(I)} -$ *Hansen* system:

$$\mathbf{r} \;=\; r\,(\mathbf{q}_1^{(I)}\cos\zeta + \mathbf{q}_2^{(I)}\sin\zeta) \quad . \tag{81}$$

Correspondingly, in the variational equations (70) and (71), for $\dot{V}$ and $V\dot{\zeta}_P$ it is possible to assume V and ζ_P, represented in analogy to polar coordinates, by the corresponding vector whose Cartesian coordinates are defined by

$$g_1 \;:=\; V\cos\zeta_P \quad , \quad g_2 \;:=\; V\sin\zeta_P \quad . \tag{82}$$

They can be determined using $r,\ \dot{r}$ and $r^2\dot{\zeta} = G$ in (38) by means of the following equations, which are valid in any general case

$$\begin{aligned} r\dot{\zeta} \;&=\; \frac{G}{r} \;=\; V\cos\left(\zeta - \zeta_P\right) \;=\; g_1\cos\zeta + g_2\sin\zeta \\ \dot{r} \;&=\; V\sin\left(\zeta - \zeta_P\right) \;=\; g_1\sin\zeta - g_2\cos\zeta \quad . \end{aligned} \tag{83}$$

Based on equations (80), we have the velocity vector in the orbit related *Hansen* system

$$\dot{\mathbf{r}} \;=\; -g_2\,\mathbf{q}_1^{(I)} + g_1\,\mathbf{q}_2^{(I)} \quad . \tag{84}$$

The derivative is

$$\ddot{\mathbf{r}} \;=\; -\dot{g}_2\,\mathbf{q}_1^{(I)} + \dot{g}_1\,\mathbf{q}_2^{(I)} - g_2\,\dot{\mathbf{q}}_1^{(I)} + g_1\,\dot{\mathbf{q}}_2^{(I)} \quad . \tag{85}$$

The *Hansen* system contains the (general) *Frenet* equations[21]

$$\begin{aligned} \dot{\mathbf{q}}_1^{(I)} \;&=\; -\dot{\eta}\,\mathbf{q}_3^{(I)}\sin\zeta \\ \dot{\mathbf{q}}_2^{(I)} \;&=\; \dot{\eta}\,\mathbf{q}_3^{(I)}\cos\zeta \\ \dot{\mathbf{q}}_3^{(I)} \;&=\; \dot{\eta}\;(\mathbf{q}_1^{(I)}\sin\zeta - \mathbf{q}_2^{(I)}\cos\zeta) \quad . \end{aligned} \tag{86}$$

Here, $\dot{\eta}$ is the rotational angular speed of the *Hansen* system with respect to the position vector ***r***. Then

$$\ddot{\mathbf{r}} \;=\; -\dot{g}_2\,\mathbf{q}_1^{(I)} + \dot{g}_1\,\mathbf{q}_2^{(I)} + V\,\dot{\eta}\cos\left(\zeta - \zeta_P\right)\mathbf{q}_3^{(I)} \quad . \tag{87}$$

Based on equation (12),

$$b_N \;=\; \dot{\eta}\,(g_1\cos\zeta \;+\; g_2\sin\zeta) \quad . \tag{88}$$

The variational equations for the parameters g_1, g_2 are derived using the definition (82) and applying the variational equations $\dot{V}$ and $V\dot{\zeta}_P$ in (70) and (71)

$$\dot{g}_1 \;=\; b_R\sin\zeta + b_T\cos\zeta \quad , \quad \dot{g}_2 \;=\; -b_R\cos\zeta + b_T\sin\zeta \quad . \tag{89}$$

Alternatively, related to the orbit angle ζ, we obtain with the radial and transversal accelerations b_R and b_T the extremely simple but generally applicable equations

$$\frac{dg_1}{d\zeta} \;=\; \frac{r^2}{G}(\;b_R\sin\zeta + b_T\cos\zeta) \;\;,\;\; \frac{dg_2}{d\zeta} \;=\; \frac{r^2}{G}(-b_R\cos\zeta + b_T\sin\zeta) \quad . \tag{90}$$

The advantage of these parameters[22] will be an elegant and symmetrical handling, thus avoiding errors in the frame of complicated „perturbation“ problems.

---

[21] see [6], Theorem 18. In [15] these formulae are derived without knowledge of the space rotation angle η

[22] Corresponding parameters as firstly introduced by *J. L. Lagrange* are widely used as “eccentricity vector” and “inclination vector”

Using the parameters $g_1 = g_1(\zeta), g_2 = g_2(\zeta)$ the radius will be represented by

$$r = \frac{G}{g_1 \cos\zeta + g_2 \sin\zeta} \quad . \tag{91}$$

The position vector will have the remarkable form

$$\mathbf{r} = G \frac{\mathbf{q}_1^{(I)} \cos\zeta + \mathbf{q}_2^{(I)} \sin\zeta}{g_1 \cos\zeta + g_2 \sin\zeta} \quad . \tag{92}$$

We must consider that this form is generally valid. In the frame of adaptation of any motion, we see that the orbit angle ζ will not explicitly be contained in the velocity vector (84). This will be a consequence of the method of adaptation because, in the case of no acceleration (i.e. traditionally speaking in the „unperturbed“ case), the velocity has to be constant. Therefore, ζ is not allowed to be contained explicitly in the velocity vector.

4 Two Instructive Examples

4.1 Adaptation of Rectilinear Motion to Motion Along a Conic Section

As a very simple but descriptive example for the adaptation of rectilinear motion to any other motion, we will consider the adaptation of a motion along a conic section.

The polar equation of a conic section is given by

$$r = \frac{p}{1 + e\cos(\zeta - \zeta_K)} \tag{93}$$

with the parameters

p = parameter of conic section [km], semilatus rectum
e = eccentricity
ζ = orbit angle [degree]
ζ_K = initial angle [degree] (pericentre angle)
$\zeta - \zeta_K$ = true anomaly [degree] .

Via *Kepler*'s third law, the parameter *p* of the conic section is related to the area parameter *G* by means of the centric gravitational constant μ by

$$G = \sqrt{\mu p} \quad . \tag{94}$$

Hence, the radial velocity will be

$$\dot{r} = \frac{G}{p} e \sin(\zeta - \zeta_K) \tag{95}$$

and the radial acceleration

$$\ddot{r} = \dot{\zeta} \frac{G}{p} e \cos(\zeta - \zeta_K) = \dot{\zeta} \frac{G}{p} e \cos(\zeta - \zeta_K) = \frac{G^2}{r^3} - \frac{G^2}{p r^2} = \frac{G^2}{r^3} - \frac{\mu}{r^2} \quad . \tag{96}$$

Comparison with the (general) *Leibniz* equation (14), gives the necessary condition for the radial acceleration in case of motion on a conic section

$$b_{RK} = -\frac{\mu}{r^2} \quad . \tag{97}$$

All other accelerations will vanish:

$$b_R =: b_{RK} + b_{R2} \quad , \quad b_{R2} = b_T = b_N = 0 \quad . \tag{98}$$

In case of the accelerations given by the equations (97) and (98), the variational equations (90) of the general rectilinear motion will be reduced to

$$\frac{dg_1}{d\zeta} = -\frac{G}{p}\sin\zeta \;,\; \frac{dg_2}{d\zeta} = \frac{G}{p}\cos\zeta \quad . \tag{99}$$

Integration gives

$$g_1 = \frac{G}{p}\cos\zeta + g_{11} \quad , \quad g_2 = \frac{G}{p}\sin\zeta + g_{21} \tag{100}$$

with the new integrational constants

$$g_{11} := g_{100}\cos\zeta_K \quad , \quad g_{21} := g_{100}\sin\zeta_K \tag{101}$$

and $g_{100} = \sqrt{g_{11}^2 + g_{21}^2}$. Consequently the radius (91) leads to

$$r = \frac{G}{g_1\cos\zeta + g_2\sin\zeta} = \frac{p}{1 + e\cos\left(\zeta - \zeta_K\right)} \tag{102}$$

with eccentricity from

$$e := \frac{p}{G}\, g_{100}. \tag{103}$$

For circular motion ($e = 0$), it must be the case that $g_{100} = 0$ since $p \neq 0, G \neq 0$ and the initial angle ζ_K will not be explained.

The equations of the parameters g_i, i.e. of the parameters V, ζ_P of the rectilinear motion, and the parameters p, e, ζ_K of the motion on a conic section, follow from

$$g_1 = V\cos\zeta_P = \frac{G}{p}\left(\cos\zeta + e\cos\zeta_K\right) \;,\; g_2 = V\sin\zeta_P = \frac{G}{p}\left(\sin\zeta + e\sin\zeta_K\right) \quad , \tag{104}$$

therefore

$$V = \frac{G}{p}\sqrt{1 + e^2 + 2e\cos\left(\zeta - \zeta_K\right)} \tag{105}$$

and

$$\cos\zeta_P = \frac{\cos\zeta + e\cos\zeta_K}{\sqrt{1 + e^2 + 2e\cos\left(\zeta - \zeta_K\right)}} \;,\; \sin\zeta_P = \frac{\sin\zeta + e\sin\zeta_K}{\sqrt{1 + e^2 + 2e\cos\left(\zeta - \zeta_K\right)}} \quad . \tag{106}$$

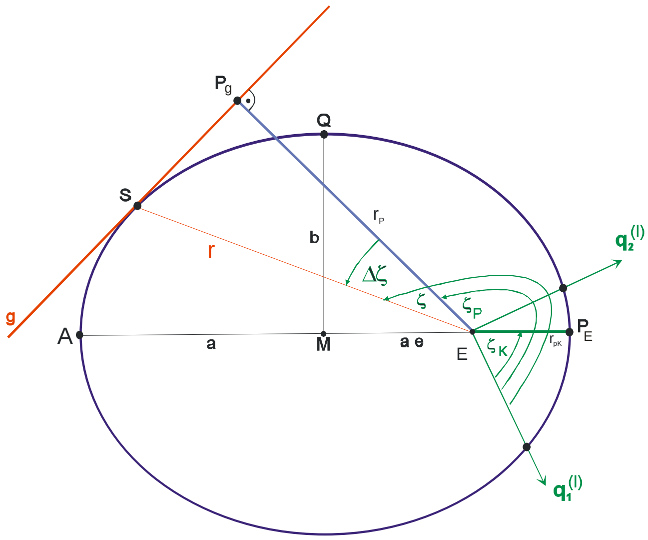

Figure 10 : Adaptation of motion along an ellipse by a straight line g. The origin E (origin of a *Hansen*-System) is centred on a focus of the ellipse. r_K is the pericentre distance of the ellipse, P_E the pericentre of the ellipse, A the apocentre of the ellipse, M the central point of the ellipse, a the semimajor axis, b the semiminor axis, e the numerical eccentricity. P_g is the pericentre of the (adaptation) straight line, r_P the pericentre distance of the adaptation line, r the radius of the moving body S, ζ the orbit angle ("first Hansen angle") of S, ζ_K the pericentre angle of the ellipse, ζ_P the pericentre angle of the straight line. $\mathbf{q}_1^{(I)}$ is the direction of the departure point of the *Hansen* system

In the case of circular motion (e = 0), these equations lead to

$$V = \frac{G}{p} = const. \quad , \quad \zeta_P = \zeta \quad , \quad (e = 0.0) \quad .$$

Consequently, the parameter ζ_P, which is constant for rectilinear motion, will be a "fast" parameter in the case of motion on a conic section.

NUMERICAL EXAMPLE: An ellipse shall be defined with semimajor axis a=10000 km and eccentricity e=0.2. Related to an (arbitrary) departure point defined by the $\mathbf{q}_1^{(I)}$ – axis of a *Hansen* system, the pericentre P of the ellipse will have the angle distance ζ_K=30°. It will be assumed that an Earth satellite is moving along such an ellipse around the Earth. Therefore, we use the geocentric gravitational constant $\mu_{♁} = 398600.440\,\mathrm{km}^3\,\mathrm{s}^{-2}$. The pericentre distance to the centre of the Earth will be $r_{PK} = a(1-e) = 8000\,\mathrm{km}$, the semilatus rectum $p = a(1-e^2) = 9600\,\mathrm{km}$. Finally, the equal area parameter has the value $G = \sqrt{\mu_{♁}\, p} = 61859.223\,\mathrm{km}^2\,\mathrm{s}^{-1}$. All these parameters have to be assumed as constants in order to calculate the parameters of an adaptation of the elliptic motion by rectilinear motion. Then, equations (103) and (101) lead to the values

$$g_{100} = e\frac{G}{p} = 1.2887\frac{\mathrm{km}}{\mathrm{s}} \quad , \quad g_{11} = \frac{g_{100}}{\cos\zeta_K} = 1.1161\frac{\mathrm{km}}{\mathrm{s}} \quad , \quad g_{21} = \frac{g_{100}}{\sin\zeta_K} = 0.6444\frac{\mathrm{km}}{\mathrm{s}} \quad .$$

The parameters g_1, g_2 of the rectilinear motion as functions of the orbit angle ζ are

$$g_1 = g_1(\zeta) = 1.1161 + 6.4437\cos\zeta \quad , \quad g_2 = g_2(\zeta) = 0.6444 + 6.4437\sin\zeta \quad .$$

Using equations (102), (105) and (106), we calculate the radius r(ζ),velocity V(ζ) and the pericentre angle ζ_P (with respect to the straight line!!!). The results for the velocity V(ζ), the radius r(ζ) and the perigee distance $r_P(\zeta) = G/V(\zeta)$ are shown in Figure 11.

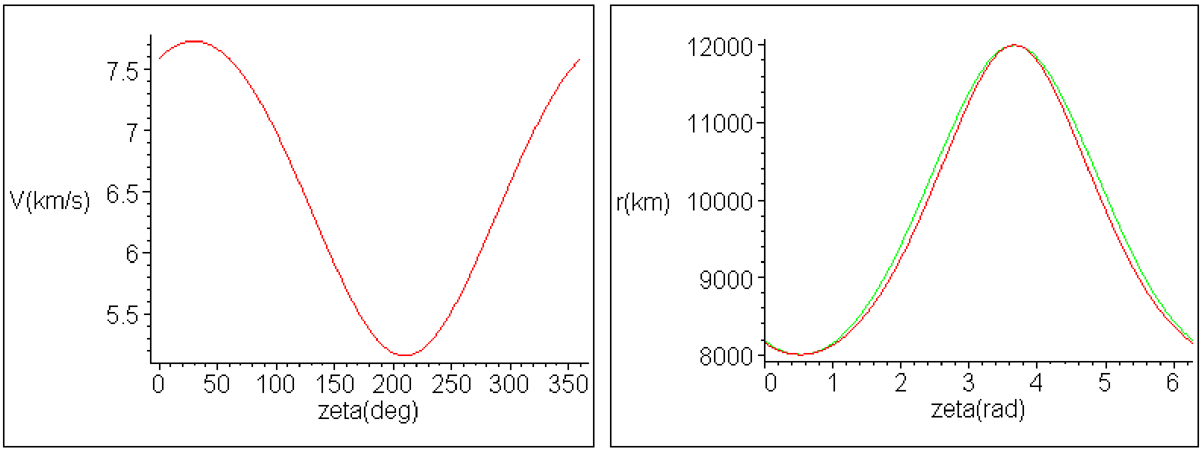

Figure 11: Course of the parameter velocity V(ζ) (left curve), the radius r(ζ) (right, upper curve, green) and the parameter perigee distance $r_P(\zeta)$ (right, lower curve, red) for one period along the ellipse, as calculated using the equations of rectilinear motion

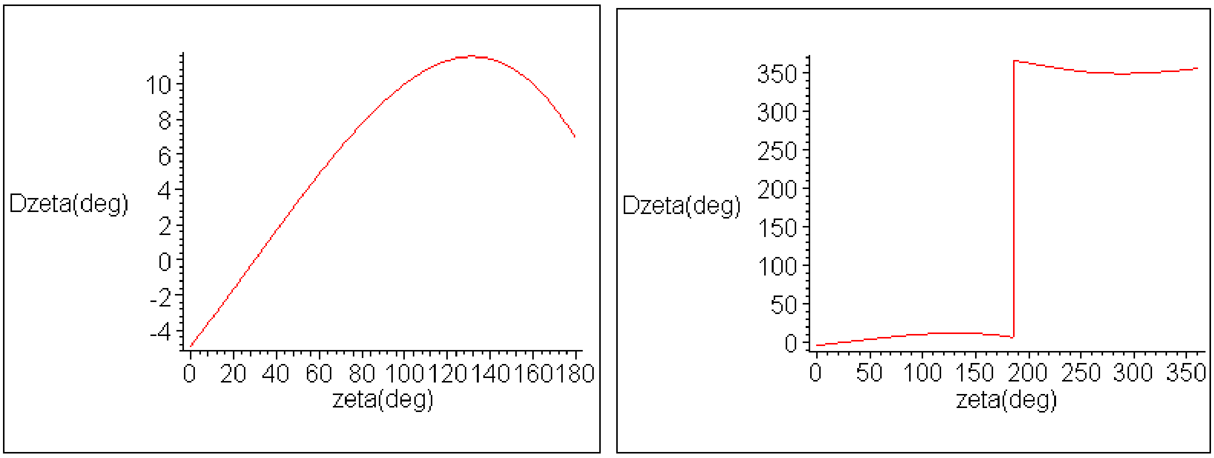

Figure 12: Course of the difference angle $\Delta\zeta = \zeta - \zeta_P$, in detail for half a period (left curve) and for the whole period (right curve)

The curves of radius r and pericentre distance r_P are nearly identical. This fact is not familiar from classical orbital mechanics, where the pericentre r_K of the ellipse is considered totally different from r_P. The adaptation curve *g* is simply the tangent at the ellipse. Figure 10 shows the relations in detail: At the osculation point *S*, the straight line *g* touches the orbital curve (here the ellipse). r is the distance of the point S from focus E of the ellipse. r_P is the distance of the point P_g on the straight line with shortest distance to the origin E. This point is the pericentre point of the straight line. Together with the motion of the straight line, this point is a function of the orbit angle ζ at the straight line. The relations will be analysed further considering the course of the orbit angle ζ_P of the pericentre point P_g on the straight line with respect to the departure point of the *Hansen* system fixed by the $\mathbf{q}_1^{(I)}$ – axis of the *Hansen* system. Figure 12 shows the difference angle $\Delta\zeta=\zeta-\zeta_P$, in dependence of the orbit angle ζ. It shows a pendulum motion of the osculating point S around the straight line perigee point P_g during one period.

Remark: conversely, an adaptation of a conic section motion to a rectilinear motion can be derived in a similar manner, showing the full equivalence between adaptation of a rectilinear motion or a conic section motion to a *Kepler*ian motion.

4.2 Radial Acceleration of an Interplanetary Spacecraft

In this example an interplanetary space probe using ion engines is considered based on the study [20]. An actual example for such a mission is the Dawn mission towards the minor planets 4/Vesta and 1/Ceres.

Leaving the Earth's orbit the space craft will have the equation of motion

$$\ddot{\mathbf{r}} = -\frac{\mu_\odot}{r^2}\mathbf{r}_0 + P_S \; AU^2 \, \frac{c_R \, A_R}{m} \, \frac{1}{r^2}\mathbf{r}_0 + \ddot{\mathbf{r}}_S \quad . \tag{107}$$

The motion will be influenced by the the solar gravitation, the solar radiation and the acceleration $\ddot{\mathbf{r}}_S$ by the ion thrusters. The following parameters are used:

$\mu_\odot = 1.32712438 \times 10^{20} \, m^3 / s^2$ is the heliocentric gravitational constant,

$P_S = 4.51 \times 10^{-6} \, \frac{kg}{m\, s^2}$ the solar pressure in the distance of one astronomical unit (AU),

c_R the solar radiation reflectivity coefficient,

$A_R[m^2]$ the equal area for solar reflectivity. In the study [20], the number A_R=113 m^2 is used.

The mass m [kg] of the spacecraft will be reduced due to mass loss triggered by the ion thrusters:

$$m = m(\zeta) = m_0 + \frac{\Delta m}{\Delta \zeta}\,\Delta\zeta \qquad , \quad \frac{\Delta m}{\Delta\zeta} \mathrel{\hat{=}} \dot{m}\,\frac{r^2}{G} \quad . \tag{108}$$

In the study 4 ion thrusters with 10 mN impuls with specific impuls $c_{eff} = 27.78\, km/s$ per thruster are used. Therefore

$$\dot{m} = \frac{40}{27.78}\frac{mN}{km/s} = 1.4398848 \times 10^{-6}\,\frac{kg}{s} \quad . \tag{109}$$

In the example, only a radial acceleration will be taken into account (i.e. b_T=b_N=0). The radial acceleration acting on the spacecraft will be

$$b_R = b_{RK} + b_{PR} + b_{SR} \quad , \tag{110}$$

where $b_{RK} = -\mu_\odot / r^2$ is the Keplerian acceleration due to the solar gravitation. The acceleration due the solar radiation will be with the abbreviation $B_S := P_S \; AE^2 \, c_R \, A_R$

$$b_{PR} =: b_{PR0} + b_{PR1} \approx \frac{B_S}{m_0}\frac{1}{r^2} - \frac{B_S}{m_0^2}\frac{1}{r^2}\frac{\Delta m}{\Delta\zeta}\Delta\zeta = \frac{B_S}{m_0}\frac{1}{r^2} - B_4\,\zeta \quad , \quad B_4 := \frac{B_S\,\dot{m}}{m_0^2\,G} \quad . \tag{111}$$

With the spacecraft's inital mass *m*=225 kg, we have the following numbers are used:

$$B_S = 1.4826859 \times 10^{10}\,\frac{kg\,km^3}{s^2} \quad , \quad B_4 = -9.466667 \times 10^{11}\,\frac{km}{s^2} \quad .$$

Assuming a constant radial impuls of the ion thrusters, the impuls is

$$b_{SR} = \frac{4\times 10}{225}\frac{mN}{kg} = 1.777778\times 10^{-7}\frac{km}{s^2} \quad .$$

Finally the variational equations (90) for the parameters of the formalism of the rectilinear motion are

$$\begin{aligned} \frac{dg_1}{d\zeta} &= \left(-\mu_\odot + \frac{B_S}{m_0} - r^2 B_4 \zeta + r^2 b_{SR}\right)\frac{\sin\zeta}{G} \\ \frac{dg_2}{d\zeta} &= -\left(-\mu_\odot + \frac{B_S}{m_0} - r^2 B_4 \zeta + r^2 b_{SR}\right)\frac{\cos\zeta}{G} \quad . \end{aligned} \tag{112}$$

The result of an analytical integration (investigated in detail in [18], chapter 4.2), with series developments using the formula manipulation program MAPLE, will be shown in Figure 13. Although the loss of mass will lead to a slow spiral motion into the outer space away from the Earth's orbit, the figure shows that radial acceleration, alone based on the technical restrici-tions of the study, will not lead to a suitable orbit within acceptable time. Therefore a tangen-tial or transversal acceleration will be recommended instead of a radial acceleration or in ad-dition to it.

In the present example devoted to radial acceleration only, we would like to estimate the or-bital behaviour for larger radial accelerations b_{SR}. In [21, formula (8.97)] a boundary estima-tion for stabil orbit is derived: the constant acceleration in radial direction must fulfil the con-dition (r_K is the circular start orbit)

$$\frac{\mu}{8 r_K^2} < b_{RC} \quad . \tag{113}$$

If this condition is not fulfilled the space craft will escape the solar system. Using the system of variational equations (112) for two constant values of the radial acceleration we obtain the results demonstrated in Figure 14. The left figure shows a stable orbit around the Sun. How-ever this orbit is far away from the usual elliptical orbit of an unperturbed motion. A small augmentation of the value of the radial acceleration will lead after some periods to the escape of the spacecraft from the solar system. It must be noted, however, that this investigation is symptomatically only. The relation (113) gives as number for the boundary acceleration for escape orbit $b_{RC} = 7.408910768\times 10^{-7}\ km/s^2$. This number is smaller than the real value ob-tained in Figure 14, because in contrast to the original consideration in [21] in the present example the elliptical path of the orbit is respected as well as the mass loss.

The computation presented in this example shows the orbital path as result, not the motion on it. For this purpose, the time as independent variable was not necessary. In the classical use of the *Leibniz* equation and the equal area law, the solution of three variationnal equations will be need, where at least the time has to be eliminated in order to get the orbital path. The pre-sent example shows that the same problem can be solved using the two symmetrical varia-tional equations (112) only. Of course, the same result can also be obtained by integration of the *Leibniz* equation (79) with the *Hansen* orbit angle as independent variable. However, the equations (112) present a more elegant description of the given problem of motion, and in mathematical context, the symmetrical form of (112) might be of some advantage to find ana-lytical solutions. In the present context, only in considering the motion of a spacecraft along its orbit the corresponding time will be calculated by solution of the equal area law (7).

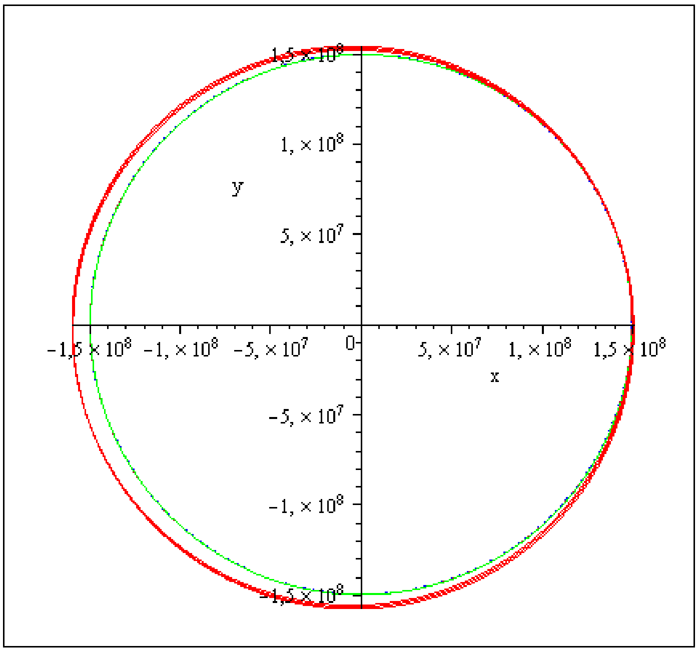

Figure 13: orbital path of interplanetary space probe with radial acceleration only due to solar attraction, solar pressure, ion thrusters. Inner (green) curve: Earth orbit used as starting curve

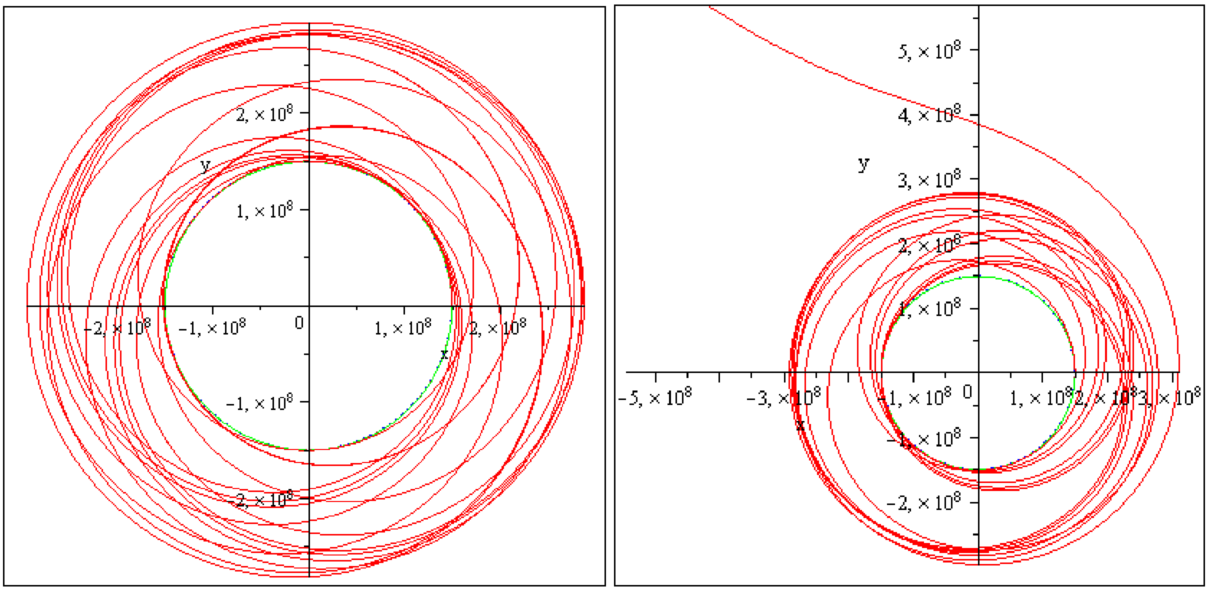

Figure 14: orbital path of interplanetary space probe with radial acceleration only due to solar attraction, solar pressure, constant impuls by ion thrusters. Left figure: stable orbit around Sun (in the center of the coordinate system) with constant radial acceleration $b_{SR} = 7.35 \times 10^{-7}\ km/s^2$, right figure: escape orbit for constant radial acceleration $b_{SR} = 7.36 \times 10^{-7}\ km/s^2$

5 Adaptation of any Motion to any other Motion

In extension of the previous considerations, a general solution of the variational equations of any motion might be solved in the following way: based on any acceleration under consideration, the variational equations are derived corresponding to this acceleration. Their solution leads to a special curve fulfilling the conditions of the given acceleration. Taking into account any other acceleration, new variational equations will be found based on the given curve and triggered by the new acceleration. The solution of these equations will lead to a new curve fulfilling the condition of the previously respected accelerations. In this manner, a step by step procedure (this is not an iteration process!) will lead to a solution of the whole given problem of motion. (This procedure will be explained in detail in a separate paper.) By a suitable application of such a procedure the target would be to obtain as far as possible an analytical solution by the aid of a modern Computer Algebra System (CAS, see e.g. [21]) before finding the final solution of the given problem by numerical procedures.

6 Conclusions

(1) Rectilinear motion can be adapted to any motion. There are no parameter singularities (besides of special cases in the relation of the motion to a fundamental basic system).

(2) If a curve along which a body is moving is given by its polar equation, and if the area parameter is known, then the acceleration producing this motion can be computed based on the general *Leibniz* equation (as demonstrated in example 4.1).

(3) A tangent at any curve is the simplest form of a tangential space: a tangential space at any primary space is known to be Euclidean even if the primary space is not *Euclide*an. Therefore, a generalization of adaptation using rectilinear motion to more general description of motions can be envisaged.

(4) With respect to a *Hansen* system, with the *Hansen* orbit angle ζ as given in the equal area law, the rectilinear motion has the radius

$$r = \frac{G}{V\cos\left(\zeta - \zeta_P\right)} \quad .$$

(5) The general distance law, with relation to a *Hansen* system with orbit angle ζ and rectilenar motion with pericentre angle $\zeta_{P,}$ has the form

$$V = \frac{G}{r}\,\frac{1}{\cos\left(\zeta - \zeta_P\right)} .$$

(6) The adaptation of any motion by means of the equations of rectilinear motion with parameters $g_1 = V\cos\zeta_P\,, g_2 = V\sin\zeta_P$, has as a consequence of the *Lagrange* constraint (66) and the *Leibniz* equation in the formulae (53) the in-plane variational equations

$$\begin{aligned} \frac{dg_1}{d\zeta} &= \frac{r^2}{G}\left(\; b_R\sin\zeta + b_T\cos\zeta\right) = \frac{G\left(\; b_R\sin\zeta + b_T\cos\zeta\right)}{\left(g_1\left(\zeta\right)\cos\zeta\; + g_2\left(\zeta\right)\sin\zeta\right)^2} \\ \frac{dg_2}{d\zeta} &= \frac{r^2}{G}\left(-b_R\cos\zeta + b_T\sin\zeta\right) = \frac{G\left(-b_R\cos\zeta + b_T\sin\zeta\right)}{\left(g_1\left(\zeta\right)\cos\zeta\; + g_2\left(\zeta\right)\sin\zeta\right)^2} \quad . \end{aligned}$$

(7) The out-of-plane variational equation of the space rotation angle η, as well as the variation of the equal area parameter G and, finally, the relation to time is given by

$$\frac{dG}{d\zeta} = \frac{r^3}{G}b_T \quad , \quad \frac{d\eta}{d\zeta} = \frac{r^3}{G^2}b_N \quad , \quad \dot{\zeta} = \frac{G}{r^2} \quad .$$

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